

Quadratic Chabauty and cubical torsors

alex.betts@cornell.edu

slides at lalexanderbetts.net

Introduction

Setup

Eventually...

- Y a smooth hyperbolic curve over a number field K .
- S a finite set of finite places of K .
- $\mathcal{Y}/\mathcal{O}_{K,S}$ regular finite type scheme with generic fibre Y .

Setup

For today:

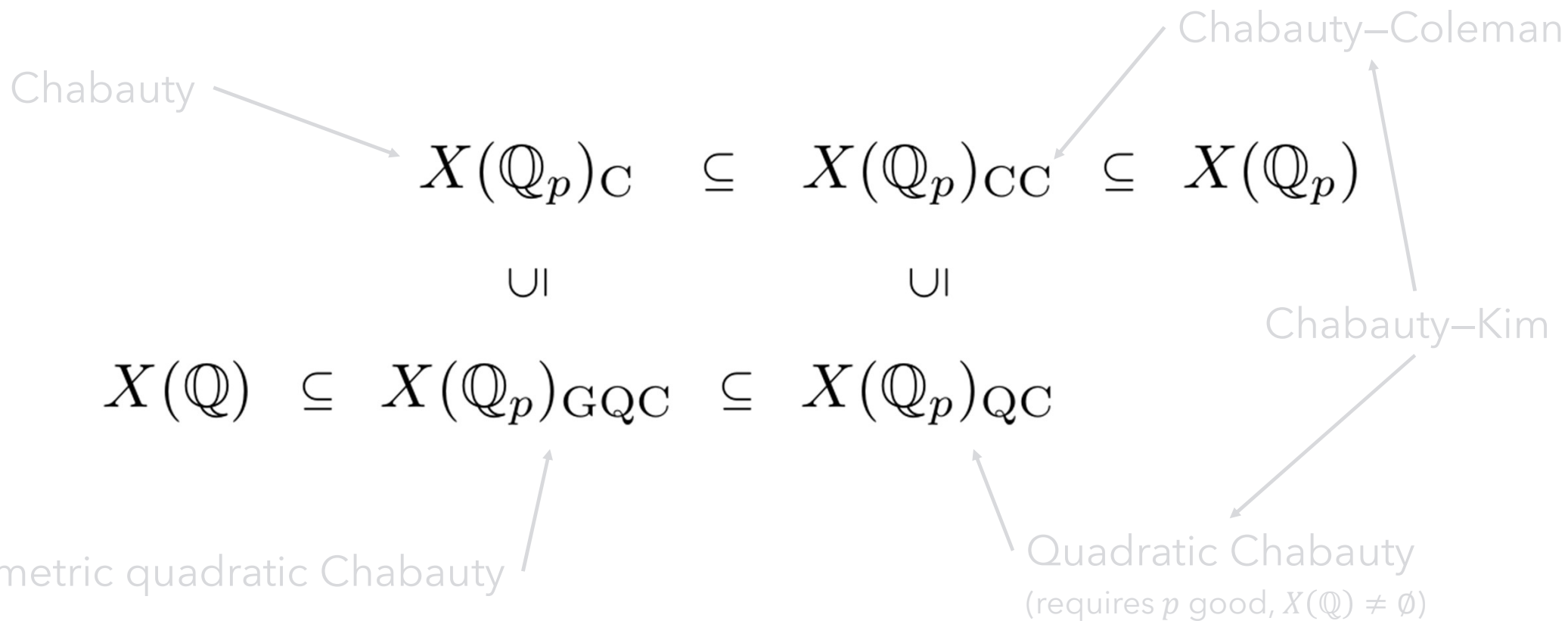
- X smooth projective over \mathbb{Q} , genus ≥ 2 .
- $X(\mathbb{Q}) \neq \emptyset$.
- \mathcal{X} smooth locus in minimal regular model.
- \mathcal{X} has geometrically connected fibres.
- $\text{rk NS}(J_{\mathbb{Q}}) = \text{rk NS}(J_{\overline{\mathbb{Q}}}) = 2$.

Chabauty methods and their loci

Variants on the Chabauty method involve studying p -adic obstruction loci:

$$X(\mathbb{Q}) \subseteq X(\mathbb{Q}_p)_{??} \subseteq X(\mathbb{Q}_p)$$

Chabauty methods and their loci

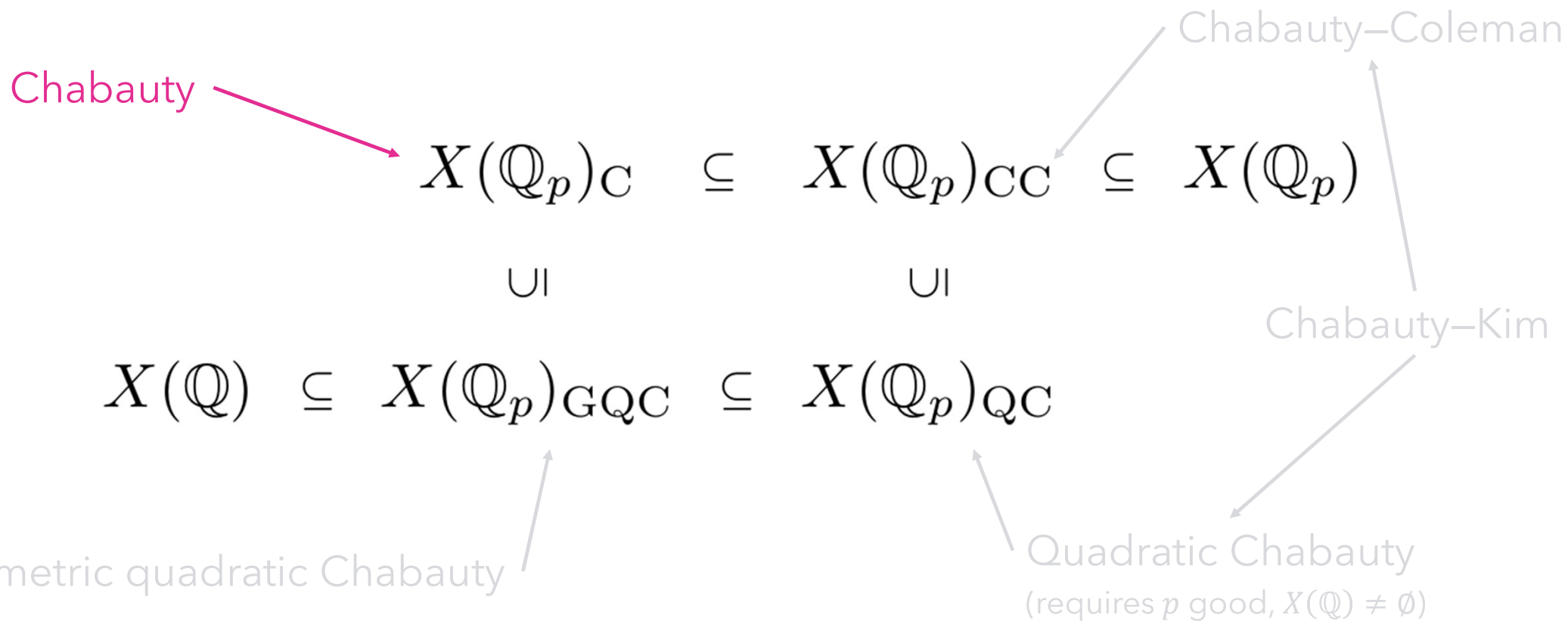


Edixhoven–Lido '21, *Geometric Quadratic Chabauty*

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Duque Rosero–Hashimoto–Spelier '23, *Geometric quadratic Chabauty and p -adic heights*

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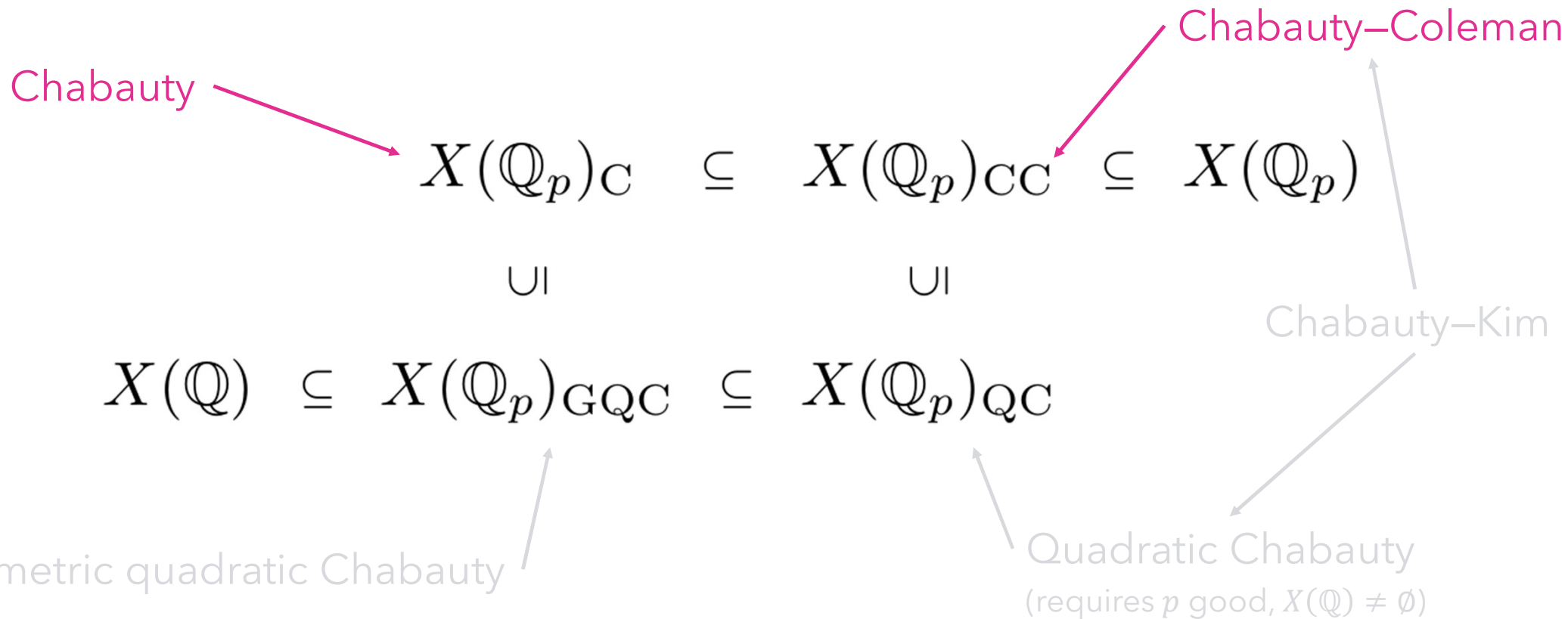


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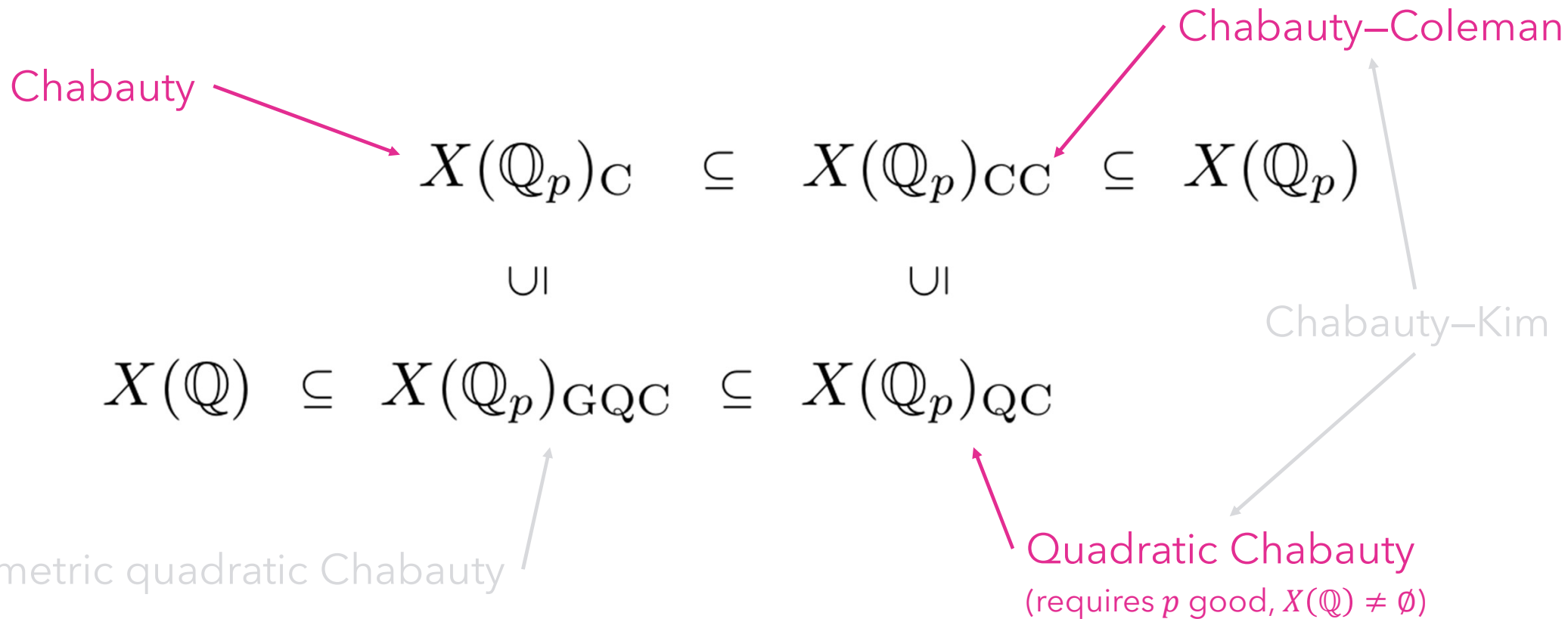


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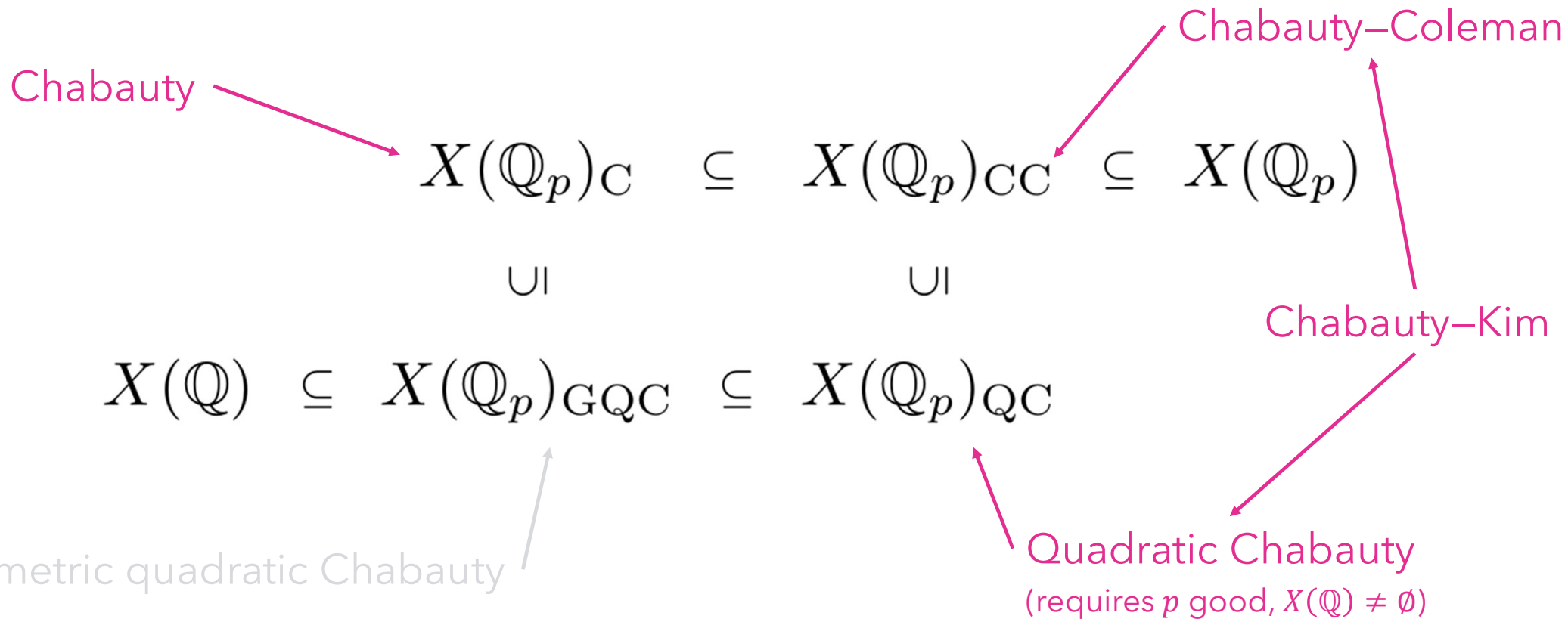


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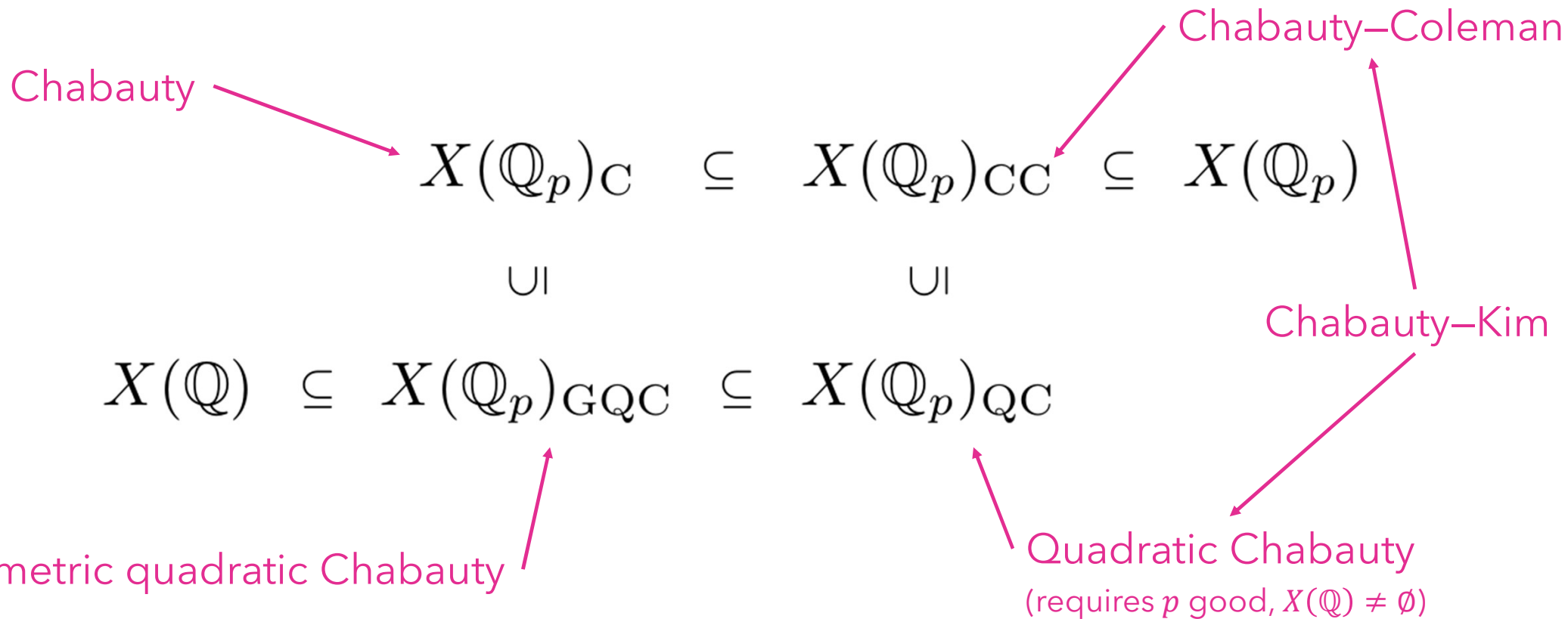


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Geometry of the QC locus

- The Chabauty and geometric quadratic Chabauty loci are defined in terms of geometry.
- So is the Chabauty–Coleman locus, using logarithms.
- The quadratic Chabauty locus is defined in terms of fundamental groups, or p -adic heights.
- Can we describe $X(\mathbb{Q}_p)_{\text{QC}}$ geometrically? Even when p is bad?

Balakrishnan–Dan-Cohen–Kim–Wewers '18, *A non-abelian conjecture of Tate–Shafarevich type*

Balakrishnan–Besser–Müller '16, *Quadratic Chabauty: p -adic height pairings and integral points on h.e. curves*

Balakrishnan–Dogra '18, *Quadratic Chabauty and rational points I: p -adic heights*

Besser–Müller–Srinivasan '25, *p -adic adelic metrics and Quadratic Chabauty I*

Abelian and quadratic Chabauty loci

Chabauty and Chabauty–Coleman

$$\begin{array}{ccccc}
 X(\mathbb{Q}) & \xhookrightarrow{\text{AJ}} & J(\mathbb{Q}) & \longrightarrow & \mathbb{Q}_p \otimes J(\mathbb{Q}) \\
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 X(\mathbb{Q}_p) & \xhookrightarrow{\text{AJ}} & J(\mathbb{Q}_p) & \xrightarrow{\log_J} & \text{Lie}(J)_{\mathbb{Q}_p}
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Chabauty locus:

$$X(\mathbb{Q}_p)_c = \text{AJ}^{-1}(\overline{J(\mathbb{Q})})$$

Chabauty–Coleman locus:

$$X(\mathbb{Q}_p)_{cc} = \text{AJ}^{-1} \log_J^{-1}(\text{im}(\text{loc}_p))$$

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Geometric quadratic Chabauty

- Lift the Abel–Jacobi embedding to a \mathbb{G}_m -torsor over the Jacobian.

$$\begin{array}{ccc} & & Q \\ & \nearrow \widetilde{AJ} & \downarrow \\ X & \xrightarrow{AJ} & J \end{array}$$

- Let \mathcal{Q} be the Néron model of Q , so \widetilde{AJ} spreads out to $\widetilde{AJ}: \mathcal{X} \rightarrow \mathcal{Q}$.
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Chabauty locus:

$$X(\mathbb{Q}_p)_{\text{GQC}} = \widetilde{\mathbf{A}}\mathbf{J}^{-1} (\overline{\mathcal{Q}_0(\mathbb{Z})})$$

Can we \mathbb{Q}_p -linearise the
right-hand side, like in
Chabauty–Coleman?

Does this recover
quadratic Chabauty?

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Cubical structures

- The variety Q does not have a group structure.
- Instead it carries a cubical structure, a trivialisation of

$$\Theta_3(Q) := \bigotimes_{I \subseteq \{1,2,3\}} m_I^* Q^{\otimes (-1)^{3-\#I}}$$

where $m_I: A^3 \rightarrow A$ is given by $m_I(x_1, x_2, x_3) = \sum_{i \in I} x_i$.

- A set (or sheaf, or variety) with a cubical structure is called a cubical torsor.
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Abelian groups vs cubical torsors

Abelian groups	Cubical torsors
Semiabelian variety	Torus-torsor over abelian torsor "AT variety"
Generalised Jacobian	Quadratic Albanese variety
Vector group (vector space)	Vectorial cubical torsor
Lie algebra $\mathrm{Lie}(A)$	Lie torsor $\mathrm{Lie}(P)$
Logarithm map $\log_A: A(\mathbb{Q}_p) \rightarrow \mathrm{Lie}(A)_{\mathbb{Q}_p}$	Logarithm map $\log_P: P(\mathbb{Q}_p) \rightarrow \mathrm{Lie}(P)_{\mathbb{Q}_p}$
Néron model	Néron model up to isogeny
Tensor product $\mathbb{Q} \otimes A$	Malčev completion $\mathbb{Q} \otimes P$

B., unpublished

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Comparison Theorem

Theorem (B. '26):

If $\text{rk}(J(\mathbb{Q})) = 0$, then $X(\mathbb{Q}_p)_{\text{IGQC}} = X(\mathbb{Q}_p)_{\text{QC}}$ for all primes p of good reduction.

- Work in progress to remove the rank zero assumption.
- In general (over \mathbb{Q}), $X(\mathbb{Q}_p)_{\text{IGQC}}$ should agree with the locus cut out by the Arakelov-theoretic quadratic Chabauty method of Besser–Müller–Srinivasan.

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Sketch of a proof (in progress)

- The embedding $\tilde{A}J: X \hookrightarrow Q$ induces a surjection on fundamental groups, and $\pi_1(Q)$ is the quotient U_T of $\pi_1(X)$ used in quadratic Chabauty.
- $\mathbb{Q}_p \otimes \mathcal{Q}_0(\mathbb{Z})$ should be the global Selmer scheme for U_T .
- $\text{Lie}(Q)_{\mathbb{Q}_p}$ should be the local Selmer scheme for U_T .
- Proofs involve constructing morphisms to Selmer schemes (universal properties) and showing that they're isomorphisms (five-lemma).
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Connection to unlikely
intersections

Quadratic torsion packets

- Assume for simplicity that Q is a *symmetric* \mathbb{G}_m -torsor over J .
- $[n]^*Q \cong Q^{\otimes n^2}$ for all n .
- $[n]: J \rightarrow J$ lifts to $[\widetilde{n}]: Q \rightarrow Q$ compatible with $[n^2]: \mathbb{G}_m \rightarrow \mathbb{G}_m$.
- An intersection of $X(\mathbb{C})$ and one fibre of the natural map
$$Q(\mathbb{C}) \rightarrow \varinjlim Q(\mathbb{C}) = \mathbb{Q} \otimes Q(\mathbb{C})$$
is called a *quadratic torsion packet*.

Theorem (B. '26, using Bertrand–Masser–Pillay–Zannier '16):

All quadratic torsion packets are finite.

Isogeny geometric QC: the rank 0 case

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If $\text{rk}(J(\mathbb{Q})) = 0$, then $\mathcal{Q}_0(\mathbb{Z})$ is finite, so $\mathbb{Q}_p \otimes \mathcal{Q}_0(\mathbb{Z}) = 1$.

$\Rightarrow X(\mathbb{Q}_p)_{\text{IGQC}}$ is the \mathbb{Q}_p -points in the intersection of a fibre of $Q(\mathbb{C}) \rightarrow \mathbb{Q} \otimes Q(\mathbb{C})$ and $X(\mathbb{C})$.

“quadratic torsion packet”
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Quadratic Stoll's Conjecture

Theorem (B. '26):

If $\text{rk}(J(\mathbb{Q})) = 0$, then there is a finite subscheme $Z \subset X$ such that:

- $Z(\mathbb{C})$ is a single quadratic torsion packet in $X(\mathbb{C})$.
- $Z(\mathbb{Q}_p) = X(\mathbb{Q}_p)_{\text{IGQC}}$ for all primes p .

Analogue of a conjecture of Stoll on the Chabauty locus.

Corollary:

In the above setup, $\#X(\mathbb{Q}_p)_{\text{QC}}$ is bounded as p ranges over all primes p of good reduction.

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Thank you for listening!

Unexpected points on once-punctured elliptic curves

joint work with Jennifer Balakrishnan

Once-punctured elliptic curves

$X = E \setminus \{\infty\}$ with E an elliptic curve of rank 0.

\Rightarrow exists a finite subscheme $Z \subset X$ with

$$X(\mathbb{Z}_p)_{\text{QC}} = Z(\mathbb{Q}_p)$$

for all primes p of good reduction.

Theorem (Bianchi, '20):

Any point in $X(\mathbb{Z}_p)_{\text{QC}}$ is torsion.

Theorem (Balakrishnan–B., in progress):

The order of any point in $X(\mathbb{Z}_p)_{\text{QC}}$ is in $\{2, 3, \dots, 10, 12, 15\}$ or > 150 .

Full-fat QC?

Lemma (B. '26):

If X is a curve of genus ≥ 4 , then the maximal 2-step unipotent quotient of the fundamental group is not the unipotent fundamental group of any smooth algebraic variety.