Weight filtrations on Selmer schemes

and effective non-abelian Chabauty

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Structures on non-abelian cohomology

A simple example

Let G be a profinite group, and let U_2 be a continuous representation of G on a $\mathbb{Q}_p$-unipotent group. Assume that U_2 is a G-equivariant central extension

$$1 \to V_2 \to U_2 \to U_1 \to 1$$

of vector groups U_1 and V_2.

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of vector groups U_1 and V_2.

Assume moreover that:

- $\mathrm{H}^0(G, U_1) = \mathrm{H}^0(G, V_2) = 0$; and
- $\mathrm{H}^1(G, U_1)$ and $\mathrm{H}^1(G, V_2)$ are finite-dimensional.

Then the functor

$$\Lambda \mapsto \mathrm{H}^1(G, U_2(\Lambda))$$

is representable by a pointed affine $\mathbb{Q}_p$-scheme $\mathrm{H}^1(G, U_2)$.

A simple example, ctd.

There is an exact sequence

$$1 \to \mathrm{H}^1(G, V_2) \to \mathrm{H}^1(G, U_2) \to \mathrm{H}^1(G, U_1) \xrightarrow{\delta} \mathrm{H}^2(G, V_2)\,,$$

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In fact, $\mathrm{H}^1(G, U_2)$ admits more structure. Suppose that $\delta = 0$, so that $\mathrm{H}^1(G, U_2)$ is a $\mathrm{H}^1(G, V_2)$-torsor over $\mathrm{H}^1(G, U_1)$. Let Θ be the $\mathrm{H}^1(G, V_2)$-torsor over $\mathrm{H}^1(G, U_1)^3$ given by

$$\Theta := \bigotimes_{I \subseteq \{1,2,3\}} m_I^* \mathrm{H}^1(G, U_2)^{(-1)^{\#I}}\,,$$

where $m_I \colon \mathrm{H}^1(G, U_1)^3 \to \mathrm{H}^1(G, U_1)$ is “sum of coordinates in I”.

A simple example, ctd.

Theorem

The torsor Θ admits a canonical trivialisation. This trivialisation defines a cubical structure on the torsor $\mathrm{H}^1(G, U_2)$, in the sense of Breen.

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Proof.

A point on Θ can be represented by eight cocycles

$$\xi_{000}, \xi_{100}, \xi_{010}, \xi_{001}, \xi_{110}, \xi_{101}, \xi_{011}, \xi_{111} \in \mathrm{Z}^1(G, U_2)$$

satisfying

$$\xi_{000} \equiv 0 \qquad \xi_{110} \equiv \xi_{100} + \xi_{010} \qquad \xi_{011} \equiv \xi_{010} + \xi_{001}$$

$$\xi_{101} \equiv \xi_{001} + \xi_{100} \qquad \xi_{111} \equiv \xi_{100} + \xi_{010} + \xi_{001}$$

modulo V_2.

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modulo V_2. One then checks that $\xi_{100}^{-1}\xi_{101}\xi_{111}^{-1}\xi_{110}\xi_{010}^{-1}\xi_{011}\xi_{001}^{-1}\xi_{000}$ is a continuous V_2-valued cocycle, and that this assignment induces a morphism $\Theta \to \mathrm{H}^1(G, V_2)$, hence a trivialisation of Θ. □

A simple example, reinterpreted

Definition

A functional $f\colon \mathrm{H}^1(G, U_2) \to \mathbb{A}^1$ is called quadratic just when it is a morphism of torsors with cubical structure, where $\mathbb{A}^1$ is viewed as the trivial $\mathbb{A}^1$-torsor with cubical structure over the point $\mathrm{Spec}(\mathbb{Q}_p)$.

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Example

Suppose that $f_1, f_2\colon \mathrm{H}^1(G, U_2) \to \mathbb{A}^1$ factor through linear maps $\mathrm{H}^1(G, U_1) \to \mathbb{A}^1$. Then the pointwise product $f_1 \cdot f_2$ is a quadratic functional.

Generalising the example: weight filtrations on cohomology

Let G be a profinite group, and let U be a continuous representation of G on a finitely generated $\mathbb{Q}_p$-pro-unipotent group. Assume that U is given with a separated G-invariant filtration

$$1 \subseteq \cdots \subseteq \mathrm{W}_{-3}U \subseteq \mathrm{W}_{-2}U \subseteq \mathrm{W}_{-1}U = U$$

by subgroup-schemes, such that $[\mathrm{W}_{-i}U, \mathrm{W}_{-j}U] \subseteq \mathrm{W}_{-i-j}U$ for all $i, j > 0$.

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- $\mathrm{H}^0(G, \mathrm{gr}^{\mathrm{W}}_{-n}U) = 0$ for all $n > 0$; and
- $\mathrm{H}^1(G, \mathrm{gr}^{\mathrm{W}}_{-n}U)$ is finite-dimensional for all $n > 0$.

Then the functor $\Lambda \mapsto \mathrm{H}^1(G, U(\Lambda))$ is representable by a pointed affine $\mathbb{Q}_p$-scheme $\mathrm{H}^1(G, U)$.

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Theorem (B.)

The W-filtration on U induces an algebra filtration $\mathrm{W}_\bullet$ on $\mathcal{O}(\mathrm{H}^1(G, U))$.

Constructing the weight filtration, 1

1) There is a natural Hopf algebra filtration on $\mathcal{O}(U)$, via the isomorphisms

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2) The functor $\mathrm{Z}^1(G,U)$ of continuous cocycles is also representable, and comes with evaluation-at-g maps

$$\mathrm{ev}_g\colon \mathrm{Z}^1(G,U) \to U$$

for each $g \in G$. We endow $\mathcal{O}(\mathrm{Z}^1(G,U))$ with the finest algebra filtration making each $\mathrm{ev}_g^*\colon \mathcal{O}(U) \to \mathcal{O}(\mathrm{Z}^1(G,U))$ filtered.

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3) $\mathcal{O}(\mathrm{H}^1(G,U))$ is a subalgebra of $\mathcal{O}(\mathrm{Z}^1(G,U))$ via the natural morphism $\mathrm{Z}^1(G,U) \to \mathrm{H}^1(G,U)$. We give $\mathcal{O}(\mathrm{H}^1(G,U))$ the subspace filtration.

Constructing the weight filtration, 2

If Λ is a W-filtered $\mathbb{Q}_p$-algebra, define

$$U(\Lambda) := \mathsf{Hom}_{\mathrm{W}-\mathsf{Alg}}(\mathcal{O}(U), \Lambda)\,,$$

which is a topological group. So we have a cohomology functor

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Remark

The proof shows that $\mathrm{H}^1(G, U)$ is non-canonically a closed subscheme of $\prod_{n>0} \mathrm{H}^1(G, \mathrm{gr}^{\mathrm{W}}_{-n} U)$, with the induced filtration. This allows us to bound $\dim \mathrm{W}_n\mathcal{O}(\mathrm{H}^1(G, U))$ in terms of the $\dim \mathrm{H}^1(G, \mathrm{gr}^{\mathrm{W}}_{-i} U)$.

Relevance to non-abelian Chabauty

The non-abelian Chabauty method

Let $\mathcal{Y}/\mathbb{Z}_S$ be a model of a hyperbolic curve $Y/\mathbb{Q}$, let $p \notin S$ be a prime of good reduction for $\mathcal{Y}$ and let $b \in \mathcal{Y}(\mathbb{Z}_S)$ be a basepoint. Let U denote a "suitable" Galois-equivariant quotient of the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of $(Y_{\bar{\mathbb{Q}}}, b)$.

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$$\begin{array}{ccccc} \mathcal{Y}(\mathbb{Z}_S) & \longrightarrow & \mathcal{Y}(\mathbb{Z}_p) & & \\ \downarrow j & & \downarrow j_p & \searrow^{j_{\mathrm{dR}}} & \\ \mathrm{Sel}_{S,U} & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1_f(G_p, U) & \xrightarrow{\sim} & U^{\mathrm{dR}}/\mathrm{F}^0 \,. \end{array}$$

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Facts:

- $\mathrm{Sel}_{S,U}$ and $\mathrm{H}^1_f(G_p, U)$ are representable by affine $\mathbb{Q}_p$-schemes, and loc_p is algebraic.
- $\mathrm{H}^1_f(G_p, U) \xrightarrow{\sim} U^{\mathrm{dR}}/\mathrm{F}^0$ is an isomorphism of $\mathbb{Q}_p$-schemes.
- j_{dR} is locally analytic with Zariski-dense image. In fact, it is explicitly given in terms of iterated Coleman integrals.

The non-abelian Chabauty method, ctd.

Consequence

Suppose that $\dim(\mathrm{Sel}_{S,U}) < \dim(\mathrm{H}^1_f(G_p, U))$. Then $\mathcal{Y}(\mathbb{Z}_S)$ is finite.

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Suppose that $\dim(\mathrm{Sel}_{S,U}) < \dim(\mathrm{H}^1_f(G_p, U))$. Then $\mathcal{Y}(\mathbb{Z}_S)$ is finite.

Proof.

The dimension inequality implies that loc_p is non-dense. Hence there is a non-zero functional $\alpha\colon U^{\mathrm{dR}}/\mathrm{F}^0 \to \mathbb{A}^1$ vanishing on the image of $\mathrm{Sel}_{S,U}$. Then $\alpha \circ j_{\mathrm{dR}}$ is locally analytic and non-zero, and vanishes on $\mathcal{Y}(\mathbb{Z}_S)$, which is thus finite.

$$
\begin{array}{ccccccc}
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\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j_p} & \overset{j_{\mathrm{dR}}}{\searrow} & & & \\
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\end{array}
$$

□

Weight filtrations and non-abelian Chabauty

In the setup of non-abelian Chabauty, U comes with a natural "weight filtration" $\mathrm{W}_\bullet$. This induces filtrations on the affine rings of $\mathrm{H}^1(G_T, U)$ and $\mathrm{H}^1(G_p, U)$, and hence on their closed subschemes $\mathrm{Sel}_{S,U}$ and $\mathrm{H}^1_f(G_p, U)$.

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Facts:

- $\mathrm{loc}_p\colon \mathrm{Sel}_{S,U} \to \mathrm{H}^1_f(G_p, U)$ is compatible with the weight filtrations on the affine rings on either side.
- The isomorphism $\mathrm{H}^1_f(G_p, U) \xrightarrow{\sim} U^{\mathrm{dR}}/\mathrm{F}^0$ is strictly compatible with the weight filtrations on the affine rings on either side.
- If $\alpha \in \mathrm{W}_n\mathcal{O}(U^{\mathrm{dR}}/\mathrm{F}^0)$, then $\alpha \circ j_{\mathrm{dR}}$ is a linear combination of iterated Coleman integrals of length $\leq n$.

Weight filtrations and non-abelian Chabauty, ctd.

Consequence

Suppose that $\dim(\mathrm{W}_n\mathcal{O}(\mathrm{Sel}_{S,U})) < \dim(\mathrm{W}_n\mathcal{O}(\mathrm{H}^1_f(G_p, U)))$. Then $\mathcal{Y}(\mathbb{Z}_S)$ is contained in the vanishing locus of a non-zero linear combination of iterated Coleman integrals of length $\leq n$.

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Proof.

The dimension inequality implies that $\mathrm{loc}_p^*\colon \mathrm{W}_n\mathcal{O}(\mathrm{H}^1_f(G, U)) \to \mathrm{W}_n\mathcal{O}(\mathrm{Sel}_{S,U})$ has non-zero kernel. Hence there is a non-zero functional $\alpha \in \mathrm{W}_n\mathcal{O}(U^{\mathrm{dR}}/\mathrm{F}^0)$ vanishing on the image of $\mathrm{Sel}_{S,U}$. Then $\alpha \circ j_{\mathrm{dR}}$ is a linear combination of iterated Coleman integrals of length $\leq n$, vanishing on $\mathcal{Y}(\mathbb{Z}_S)$. □

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Remark

Using the "nice differential operators" machinery of Balakrishnan–Dogra, should be possible to bound the number of zeroes of $\alpha \circ j_{\mathrm{dR}}$ in terms of n.

Example: the thrice-punctured line

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Let's take S a set of $s = \#S$ primes, and $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}_S} \setminus \{0, 1, \infty\}$. Fix a choice of cusp $\Sigma_v \in \{0, 1, \infty\}$ for each $v \in S$, and define

$$\mathcal{Y}(\mathbb{Z}_S)^\Sigma := \{z \in \mathcal{Y}(\mathbb{Z}_S) : z \in \mathcal{Y}(\mathbb{Z}_v) \text{ or } z \text{ reduces to } \Sigma_v \bmod v, \forall v \in S\} .$$

Thus, $\mathcal{Y}(\mathbb{Z}_S)$ is the union of the 3^s sets $\mathcal{Y}(\mathbb{Z}_S)^\Sigma$.

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Thus, $\mathcal{Y}(\mathbb{Z}_S)$ is the union of the 3^s sets $\mathcal{Y}(\mathbb{Z}_S)^\Sigma$.

For each suitable quotient U of the fundamental group, there is a corresponding "refined Selmer scheme" $\mathrm{Sel}_{\Sigma,U} \subseteq \mathrm{H}^1(G_T, U)$ containing the image of $\mathcal{Y}(\mathbb{Z}_S)^\Sigma$ under the non-abelian Kummer map j.

$\#S = 2$, depth 2

Take U_2 the 2-step unipotent quotient of the fundamental group. Then

$$\dim \mathrm{W}_{2n}\mathcal{O}(\mathsf{H}^1_f(G_p, U_2)) = n\text{th coeff. of } (1-t)^{-3}\cdot(1-t^2)^{-1}$$
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So if $s=2$, then $\dim \mathrm{W}_4\mathcal{O}(\mathrm{Sel}_{\Sigma,U_2}) < \dim \mathrm{W}_4\mathcal{O}(\mathrm{H}^1_f(G_p, U_2))$, and so $\mathcal{Y}(\mathbb{Z}_S)^\Sigma$ is contained in the vanishing locus of a non-zero linear combination of multiple polylogarithms of length ≤ 2.

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Theorem (Best–B.–Kumpitsch–Lüdtke–McAndrew–Qian–Studnia–Xu)

Suppose that $S = \{\ell, q\}$, and that $\Sigma_\ell = 1$ and $\Sigma_q = 0$. Then $\mathcal{Y}(\mathbb{Z}_S)^\Sigma$ is contained in the locus cut out by the equation

$$a_{\ell,q}\mathrm{Li}_2(z) = a_{q,\ell}\mathrm{Li}_2(1-z)\,,$$

where $a_{\ell,q}, a_{q,\ell} \in \mathbb{Q}_p$ are the coefficients computed by Dan-Cohen–Wewers.

$\#S$ arbitrary, infinite depth

Take U the whole $\mathbb{Q}_p$-pro-unipotent fundamental group. Then

$$\dim \mathrm{W}_{2n}\mathcal{O}(\mathsf{H}^1_f(G_p, U)) = n\text{th coeff. of } (1-t)^{-1}\cdot(1-2t)^{-1}$$

$$\dim \mathrm{W}_{2n}\mathcal{O}(\mathrm{Sel}_{\Sigma,U}) \leq n\text{th coeff. of } (1-t)^{-s-1}\cdot \prod_{i\geq 3,\text{odd}}(1-t^i)^{-M(2,i)}$$

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This implies that there is some $n \leq 4^s$ such that $\dim \mathrm{W}_{2n}\mathcal{O}(\mathrm{Sel}_{\Sigma,U}) < \dim \mathrm{W}_{2n}\mathcal{O}(\mathrm{H}^1_f(G,U))$, and hence $\mathcal{Y}(\mathbb{Z}_S)^\Sigma$ is contained in the vanishing locus of a non-zero linear combination of multiple polylogarithms of length $\leq 4^s$.

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Theorem (B.)

$\#\mathcal{Y}(\mathbb{Z}_S) \leq 16 \cdot 6^s \cdot 2^{4^s}$.

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$$\dim \mathrm{W}_{2n}\mathcal{O}(\mathrm{Sel}_{\Sigma,U}) \leq n\text{th coeff. of } (1-t)^{-s-1} \cdot \prod_{i\geq 3,\text{odd}} (1-t^i)^{-M(2,i)}$$

This implies that there is some $n \leq 4^s$ such that $\dim \mathrm{W}_{2n}\mathcal{O}(\mathrm{Sel}_{\Sigma,U}) < \dim \mathrm{W}_{2n}\mathcal{O}(\mathrm{H}^1_f(G, U))$, and hence $\mathcal{Y}(\mathbb{Z}_S)^\Sigma$ is contained in the vanishing locus of a non-zero linear combination of multiple polylogarithms of length $\leq 4^s$.

Theorem (B.)

$\#\mathcal{Y}(\mathbb{Z}_S) \leq 16 \cdot 6^s \cdot 2^{4^s}$.

Theorem (Evertse)

$\#\mathcal{Y}(\mathbb{Z}_S) \leq 3 \cdot 7^{2s+3}$.