Lecture 10: Bloch–Kato Selmer schemes

We will be especially interested in cohomology schemes when the profinite group G is the absolute Galois group of a local or global field. In this case, the cohomology schemes come with certain distinguished closed subschemes known as the Bloch–Kato Selmer schemes. We will introduce these in three different settings:

- local Bloch–Kato Selmer schemes ($\ell \neq p$);
- local Bloch–Kato Selmer schemes ($\ell = p$);
- global Bloch–Kato Selmer schemes.

But before we get to these, let us review the theory of Bloch–Kato Selmer groups, the abelian theory which we are trying to generalize here. For an excellent exposition, see Joel Bellaïche's notes from the CMI summer school in Honolulu.

Bloch–Kato Selmer groups

Let K be a finite extension of $\mathbb{Q}_\ell$ and let V be a $\mathbb{Q}_p$-linear representation of G_K. We explikitly allow the possibility that $\ell = p$. In their paper, *L-functions and Tamagawa numbers of motives*, Spencer Bloch and Kazuya Kato studied the structure of the local cohomology groups

$$H^1(G_K, V).$$

More precisely, they defined certain subspaces $H^1_*(G_K, V)$ (for various values of $*$), whose definitions are purely representation-theoretic, but which are usually of some geometric significance when the representation V arises from geometry.

Let's begin by discussing the case $\ell \neq p$. Here, the only interesting subspace of the cohomology is the unramified part, i.e.

$$H^1_{ur}(G_K, V) := \ker\left(H^1(G_K, V) \longrightarrow H^1(I_K, V)\right)$$

where $I_K \trianglelefteq G_K$ is the ~~ramif~~ inertia subgroup.

Remark: If you like thinking of $H^1(G_K, V)$ as the Ext-group $\mathrm{Ext}^1(\mathbb{1}, V)$ in the category of G_K-representations, then $H^1_{ur}(G_K, V)$ has a particularly nice interpretation. If V is an unramified representation, then $H^1_{ur}(G_K, V)$ is just the space of unramified extensions

$$0 \to V \to E \to \mathbb{1} \to 0. \qquad \circledast$$

In general, $H^1_{ur}(G_K, V)$ parametrises those extensions $\circledast$ which are split I_K-equivariantly, or equivalently those extensions $\circledast$ for which the sequence

$$0 \to V^{I_K} \to E^{I_K} \to \mathbb{1} \to 0$$

is still exact.

There are many interesting things to say about the unramified cohomology (e.g. that $H^1_{ur}(G_K, V)$ and $H^1_{ur}(G_K, V^*(1))$ are exact annihilators under the Poincaré duality pairing for local cohomology) but we'll only really use one fact about them in what follows.

Lemma: Suppose that V is pure of negative weight (or that it is mixed with negative weights). Then

$$H^1_{ur}(G_K, V) = 0.$$

Proof: One can check that the condition on the weights implies that $V^{G_K} = 0$. Now the inflation-restriction exact sequence

$$0 \to H^1(\hat{\mathbb{Z}}, V^{I_K}) \to H^1(G_K, V) \longrightarrow H^1(I_K, V)^{\hat{\mathbb{Z}}}$$

gives us identifications

$$H^1_{ur}(G_K, V) = H^1(\hat{\mathbb{Z}}, V^{I_K}) = \frac{V^{I_K}}{(\varphi - 1)V^{I_K}}$$

where φ is a topological generator. Since $(V^{I_K})^{\varphi} = V^{G_K} = 0$, we know that $\varphi - 1 : V^{I_K} \to V^{I_K}$ is injective, hence surjective, so $H^1_{ur}(G_K, V) = 0$. $\square$

In the $\ell=p$ case, the theory is much richer, and uses Fontaine's period rings B_{cris} and B_{dR}. We shall neither give nor need the precise definitions of these period rings. Suffice it to say that B_{dR} is a topological $\overline{K}$-algebra with a continuous action of G_K extending the usual action on $\overline{K}$, and B_{cris} is a topological $\mathbb{Q}_p^{ur}$-algebra with a continuous action of G_K extending the usual action on $\mathbb{Q}_p^{ur}$. Moreover B_{cris} is a topological subalgebra of B_{dR} in a manner compatible with their G_K-actions. There is also a semilinear "crystalline Frobenius" φ on B_{cris}, ~~and~~ commuting with the action of G_K, so that we also have the G_K-invariant subring $B_{cris}^{\varphi=1} \subset B_{cris}$.

<u>Definition:</u> Let V be a de Rham representation of G_K. Then define

$$H^1_e(G_K,V) := \ker\left(H^1(G_K,V) \longrightarrow H^1(G_K, B_{cris}^{\varphi=1} \otimes_{\mathbb{Q}_p} V)\right)$$

$$H^1_f(G_K,V) := \ker\left(H^1(G_K,V) \longrightarrow H^1(G_K, B_{cris} \otimes_{\mathbb{Q}_p} V)\right)$$

$$H^1_g(G_K,V) := \ker\left(H^1(G_K,V) \longrightarrow H^1(G_K, B_{dR} \otimes_{\mathbb{Q}_p} V)\right).$$

Since $B_{cris}^{\varphi=1} \subseteq B_{cris} \subseteq B_{dR}$, we have

$$0 \subseteq H^1_e(G_K,V) \subseteq H^1_f(G_K,V) \subseteq H^1_g(G_K,V) \subseteq H^1(G_K,V).$$

<u>Remark</u>: In the definition of H^1_g, one can equally replace B_{dR} with the semistable period ring B_{st}.

Roughly speaking, the expectation in the $\ell = p$ case is that the quotients

$$\frac{H^1_f(G_K, V)}{H^1_e(G_K, V)} \quad \text{and} \quad \frac{H^1_g(G_K, V)}{H^1_e(G_K, V)}$$

behave similarly to the groups

$$H^1_{ur}(G_K, V) \quad \text{and} \quad H^1(G_K, V)$$

in the $\ell \neq p$ case. For instance, one has the following.

<u>Lemma</u>: Suppose that V is pure of negative weight (or that V is mixed with negative weights). Then

$$H^1_f(G_K, V) = H^1_e(G_K, V)$$

<u>Proof</u>: The map $\varphi - 1 : B_{cris} \longrightarrow B_{cris}$ is surjective (see the proof of Proposition 1.17 in Bloch-Kato's paper) so we have a short exact sequence

$$0 \longrightarrow B_{cris}^{\varphi=1} \longrightarrow B_{cris} \xrightarrow{\varphi - 1} B_{cris} \longrightarrow 0$$

of topological $\mathbb{Q}_p$-algebras with G_K-action

Tensoring with V gives

$$0 \to B_{cris}^{\varphi=1} \otimes V \longrightarrow B_{cris} \otimes V \xrightarrow{\varphi-1} B_{cris} \otimes V \to 0.$$

Taking the LES in cohomology* gives

$$D_{cris}(V) \xrightarrow{\varphi-1} D_{cris}(V) \to H^1(G_K, B_{cris}^{\varphi=1} \otimes V) \to H^1(G_K, B_{cris} \otimes V)$$

so the kernel of $H^1(G_K, B_{cris}^{\varphi=1} \otimes V) \to H^1(G_K, B_{cris} \otimes V)$ is $D_{cris}(V)/(\varphi-1)$. The assumption on the weights ensures that φ acts without fixed points on $D_{cris}(V)$, so $\varphi - 1$ is surjective and $H^1(G_K, B_{cris}^{\varphi=1} \otimes V) \to H^1(G_K, B_{cris} \otimes V)$ is injective. This implies that

$$H^1_e(G_K, V) = H^1_f(G_K, V). \qquad \square$$

* Actually, I don't know whether this sequence is topologically split, but this part of the long exact sequence exists anyway.

The Bloch-Kato Selmer group H^1_e can be understood quite explicitly thanks to a construction known as the <u>Bloch-Kato exponential</u>. For this, if V is a de Rham representation of G_K, we define

$$D_{dR}(V) := (B_{dR} \otimes V)^{G_K}$$

$$D_{dR}^+(V) := (B_{dR}^+ \otimes V)^{G_K}$$

$$D_{cris}^{\varphi=1}(V) := (B_{cris}^{\varphi=1} \otimes V)^{G_K}$$

where $B_{dR}^+ \subseteq B_{dR}$ is the 0^{th} step of the Hodge filtration on B_{dR}.

<u>Proposition</u>: Inside B_{dR}, we have $B_{cris}^{\varphi=1} \cap B_{dR}^+ = \mathbb{Q}_p$ (with its usual topology) and $B_{cris}^{\varphi=1} + B_{dR}^+ = B_{dR}$.

Put another way, we have an exact sequence

$$(*) \quad 0 \longrightarrow \mathbb{Q}_p \longrightarrow B_{cris}^{\varphi=1} \oplus B_{dR}^+ \longrightarrow B_{dR} \longrightarrow 0$$

$$x \longmapsto (x, x)$$

$$(x, y) \longmapsto y - x.$$

~~Tensoring $(*)$ with our representation V and taking Galois cohomology gives the Bloch-Kato exponential exact sequence~~

$$\cancel{0 \to V^{G_K} \longrightarrow D_{cris}^{\varphi=1}(V) \oplus D_{dR}^+(V) \to D_{dR}(V) \to H^1(G_K, V)}$$

Lemma (Bloch-Kato) For V de Rham, the map

$$H^1(G_K, B_{dR}^+ \otimes V) \longrightarrow H^1(G_K, B_{dR} \otimes V)$$

is injective.

Consequence:

$$H^1_e(G_K, V) = \ker\left(H^1(G_K, V) \to H^1(G_K, B_{cris}^{\varphi=1} \otimes V) \oplus H^1(G_K, B_{dR}^+ \otimes V)\right)$$

Proof: the kernel is clearly contained in H^1_e.
Conversely, if $\xi \in H^1_e(G_K, V)$, then its image in $H^1(G_K, B_{cris}^{\varphi=1} \otimes V) \oplus H^1(G_K, B_{dR}^+ \otimes V)$ is of the form $(0, \alpha)$. The image of α in $H^1(G_K, B_{dR} \otimes V)$ is 0, so $\alpha = 0$ by the Lemma. So we are done. $\square$

So if we take $\circledast$, tensor with V and take $H^0(G_K, -)$, we get the Bloch-Kato exponential exact sequence

$$0 \to V^{G_K} \to D_{cris}^{\varphi=1}(V) \oplus D_{dR}^+(V) \to D_{dR}(V) \longrightarrow H^1_e(G_K, V) \to 0,$$

where the coboundary map $D_{dR}(V) \longrightarrow H^1_e(G_K, V)$ is known as the Bloch-Kato exponential.

Remark: the B-K exponential sequence is usually written

$$0 \to V^{G_K} \to D_{cris}^{\varphi=1}(V) \longrightarrow \frac{D_{dR}(V)}{D_{dR}^+(V)} \to H^1_e(G_K, V) \to 0.$$

Global Bloch–Kato Selmer groups

One can also use similar ideas to cut out interesting subspaces of cohomology groups $H^1(G_K, V)$ where K is a number field (and the representation V is de Rham at all p-adic places of K). It is convenient to have some flexibility in what local conditions we impose, as follows.

Definition: A Selmer structure $\mathfrak{S}$ for V is a choice of a subspace $\mathfrak{S}_v \leq H^1(G_v, V)$ for all finite places v of K, such that $\mathfrak{S}_v = H^1_{ur}(G_v, V)$ for all but finitely many v.

decomposition group

The corresponding Selmer group

$$\mathrm{Sel}_{\mathfrak{S},V} \leq H^1(G_K, V)$$

is defined to be the set of cohomology classes $[\xi] \in H^1(G_K, V)$ such that $[\xi]|_{G_v} \in \mathfrak{S}_v$ for all finite places v.

<u>Example:</u> Take the Selmer structure $\mathfrak{S}$ given by

$$\mathfrak{S}_v := \begin{cases} H^1_{ur}(G_v, V) & \text{if } v \nmid p \\ H^1_f(G_v, V) & \text{if } v \mid p. \end{cases}$$

The corresponding Selmer group is denoted

$$H^1_f(G_K, V).$$

More generally, fix a finite set S of finite places of K and define a Selmer structure $\mathfrak{S}$ by

$$\mathfrak{S}_v := \begin{cases} H^1_{ur}(G_v, V) & \text{if } v \nmid p,\ v \notin S \\ H^1(G_v, V) & \text{if } v \nmid p,\ v \in S \\ H^1_f(G_v, V) & \text{if } v \mid p,\ v \notin S \\ H^1_g(G_v, V) & \text{if } v \mid p,\ v \in S. \end{cases}$$

The resulting Selmer group is denoted

$$H^1_{f,S}(G_K, V).$$

<u>Remark:</u> the Selmer ~~struc~~ group $Sel_{\mathfrak{S},V}$ associated to a Selmer structure $\mathfrak{S}$ is always finite-dimensional, even though $H^1(G_K, V)$ need not be. (E.g. $H^1(G_K, \mathbb{Q}_p(1)) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n (K^\times / K^{\times p^n})$ by Kummer theory, which is infinite-dimensional.)

Bloch–Kato Selmer schemes

Let's now explain how to generalise all of these constructions to the non-abelian (pro-unipotent) case.

Local Bloch–Kato Selmer schemes: $\ell \neq p$

To begin with, suppose that $K/\mathbb{Q}_\ell$ is a finite extension, where $\ell \neq p$. Let $U/\mathbb{Q}_p$ be a finitely generated pro-unipotent group endowed with a continuous action of G_K. Suppose moreover that U possesses a separated G_K-stable filtration

$$U = W_{-1}U \trianglerighteq W_{-2}U \trianglerighteq \dots$$

such that $V_n := W_{-n}U / W_{-n-1}U$ is pure of weight $-n$ for all $n \geq 1$.

Since G_K has property (F), ~~and~~ we know that $H^i(G_K, V_n)$ is finite-dimensional for all n and i. Moreover, $H^0(G_K, V_n) = 0$ for all n by the weight condition. So the cohomology functor

$H^1(G_K, U)$ is representable.

We can define a subfunctor $H^1_{ur}(G_K, U) \subseteq H^1(G_K, U)$ by

$$H^1_{ur}(G_K, U)(\Lambda) := \ker\left(H^1(G_K, U(\Lambda)) \to H^1(I_K, U(\Lambda))\right)$$

for $\Lambda \in \underrightarrow{\mathrm{Alg}}_{\mathbb{Q}_p}$. We claim that $H^1_{ur}(G_K, U)$ is representab[le] by a closed subscheme of $H^1(G_K, U)$. In fact, this true for kinda trivial reasons.

<u>Proposition:</u> Under the above assumptions,

$$H^1_{ur}(G_K, U) = \{*\}$$

is the distinguished point of $H^1(G_K, U)$, viewed as a one-point scheme in the usual way.

<u>Proof</u>: There is a non-abelian inflation-restriction exact sequence

$$1 \to H^1(\hat{\mathbb{Z}}, U(\Lambda)^{I_K}) \to H^1(G_K, U(\Lambda)) \to H^1(I_K, U(\Lambda))$$

which gives an isomorphism

$$H^1_{ur}(G_K, U) \cong H^1(\hat{\mathbb{Z}}, U^{I_K})$$

of functors, where U^{I_K} is the I_K-fixed subgroup-scheme. Endowing U^{I_K} with the induced weight filtrati[on] from U, we have

$$gr^W_{-n}(U^{I_K}) \subseteq gr^W_{-n}(U)^{I_K} = V_n^{I_K}, \text{ so}$$

$$[gr^W_{-n}(U^{I_K})]^{\hat{\mathbb{Z}}} = V_n^{G_K} = 0$$

So U^{I_K} satisfies the conditions of the previous lecture, so that $H^1(\hat{\mathbb{Z}}, U^{I_K})$ is representable by a closed subscheme of

$$\prod_n H^1(\hat{\mathbb{Z}}, gr_n^W(U^{I_K})) = 0$$

~~But since $H^1(\hat{\mathbb{Z}}$~~

since $\dim_{\mathbb{Q}_p} H^1(\hat{\mathbb{Z}}, gr_n^W(U^{I_K})) = \dim_{\mathbb{Q}_p} H^0(\hat{\mathbb{Z}}, gr_n^W(U^{I_K}))$
$= 0$.

Hence $H^1_{ur}(G_K, U) = \{*\}$ as claimed $\square$

Local Bloch-Kato Selmer schemes: $\ell = p$

Now suppose that $K/\mathbb{Q}_p$ is a finite extension. Let $U/\mathbb{Q}_p$ be a finitely generated pro-unipotent group endowed with a continuous de Rham action of G_K. Suppose moreover that U is ~~de Rham and~~ endowed with a separated G_K-stable filtration

$$U = W_{-1}U \trianglerighteq W_{-2}U \trianglerighteq \ldots$$

such that $V_n := W_{-n}U/W_{-n-1}U$ is pure of weight $-n$ for all $n \geq 1$. These conditions again imply that the cohomology functor $H^1(G_K, U)$ is representable

We define closed subschemes

$$\{*\}\subseteq H^1_e(G_K,U)\subseteq H^1_f(G_K,U)\subseteq H^1_g(G_K,U)\subseteq H^1(G_K,U)$$

as follows.

$B^{\varphi=1}_{cris}\otimes\Lambda$ is topologised like $(B^{\varphi=1}_{cris})^{\oplus I}$; the topology on $U(B^{\varphi=1}_{cris}\otimes\Lambda)$ is then the natural one

<u>Definition</u>: For $\Lambda\in\underline{\mathrm{Alg}}_{\mathbb{Q}_p}$ define

$$H^1_e(G_K,U(\Lambda)):=\ker\left(H^1(G_K,U(\Lambda))\to H^1(G_K,U(B^{\varphi=1}_{cris}\otimes\Lambda))\right)$$

$$H^1_f(G_K,U(\Lambda)):=\ker\left(H^1(G_K,U(\Lambda))\to H^1(G_K,U(B_{cris}\otimes V))\right)$$

$$H^1_g(G_K,U(\Lambda)):=\ker\left(H^1(G_K,U(\Lambda))\to H^1(G_K,U(B_{dR}\otimes V))\right)$$

This defines pointed subfunctors

$$H^1_e(G_K,U)\subseteq H^1_f(G_K,U)\subseteq H^1_g(G_K,U)\subseteq H^1(G_K,U).$$

<u>Theorem</u>: The above subfunctors are representable by closed ~~subfunctors~~ subschemes of $H^1(G_K,U)$.

For the proof, we fix some notation. For $B\in\{B^{\varphi=1}_{cris},B_{cris},B_{dR}\}$, we let U_B denote the functor

$$\Lambda\longmapsto U(B\otimes\Lambda)$$

from $\mathbb{Q}_p$-algebras to topological groups with continuous G_K-action. We define $U_{n,B}$ similarly, where $U_n=U/W_{-n-1}U$.

We also define functors

$$D_{cris}^{\varphi=1}(U): \underline{\mathrm{Alg}}_{\mathbb{Q}_p} \longrightarrow \underline{\mathrm{Grp}}$$

$$D_{cris}(U): \underline{\mathrm{Alg}}_{K_0} \longrightarrow \underline{\mathrm{Grp}}$$

$$D_{dR}(U): \underline{\mathrm{Alg}}_{K} \longrightarrow \underline{\mathrm{Grp}}$$

by

$$D_{cris}^{\varphi=1}(U)(\Lambda) := U(B_{cris}^{\varphi=1} \otimes_{\mathbb{Q}_p} \Lambda)^{G_K}$$

$$D_{cris}(U)(\Lambda) := U(B_{cris} \otimes_{K_0} \Lambda)^{G_K}$$

$$D_{dR}(U)(\Lambda) := U(B_{dR} \otimes_{K} \Lambda)^{G_K}.$$

Here we use that $(B_{cris}^{\varphi=1})^{G_K} = \mathbb{Q}_p$, $(B_{cris})^{G_K} = K_0$ is the maximal unramified subfield of K, and $(B_{dR})^{G_K} = K$.

<u>Proposition</u>: $D_{cris}^{\varphi=1}(U)$, $D_{cris}(U)$ and $D_{dR}(U)$ are representable by pro-unipotent groups over $\mathbb{Q}_p$, K_0 and K, respectively. Moreover, under our assumptions above, we have $D_{cris}^{\varphi=1}(U) = 1$.

<u>Proof</u>: We use the logarithm isomorphism

$$U \cong \mathrm{Lie}(U).$$

We have

$$D_{dR}(U)(\Lambda) = U(B_{dR}\otimes_K \Lambda)^{G_K} \cong \left(\mathrm{Lie}(U)_{B_{dR}\otimes_K \Lambda}\right)^{G_K}$$

$$= \left(\mathrm{Lie}(U)_{B_{dR}}\right)^{G_K}_{\Lambda} = D_{dR}(\mathrm{Lie}(U))_{\Lambda}$$

So $D_{dR}(U)$ is representable by the affine space associated to $D_{dR}(\mathrm{Lie}(U)) \in \text{pro-}\underline{\mathrm{Vec}}_K$. One can check that the above isomorphism identifies the group law on $D_{dR}(U)$ with the Baker–Campbell-Hausdorff product on $\mathbb{A}(D_{dR}(\mathrm{Lie}(U)))$, and so $D_{dR}(U) \cong \mathbb{G}(D_{dR}(\mathrm{Lie}(U)))$ is pro-unipotent.

A similar argument shows that

$$D_{cris}(U) \cong \mathbb{G}(D_{cris}(\mathrm{Lie}(U))) \quad \text{and}$$

$$D_{cris}^{\varphi=1}(U) \cong \mathbb{G}(D_{cris}^{\varphi=1}(\mathrm{Lie}(U)))$$

are both representable by pro-unipotent groups. The final assertion follows from the assumption on the weights of U: we have $D_{cris}^{\varphi=1}(\mathrm{Lie}(U)) = 0$ so $D_{cris}^{\varphi=1}(U) = 1$.

Now we return to the proof of the theorem, beginning with the proof of representability of

$$H^1_e(G_K, U) = \ker\left(H^1(G_K, U) \longrightarrow H^1(G_K, U_{B_{\mathrm{cris}}^{\varphi=1}})\right).$$

<u>Step 1</u>: $H^1_e(G_K, U_n)$ is representable by a closed subscheme of $H^1(G_K, U_n)$ for all n.

<u>Proof</u>: We proceed by induction, starting from the trivial base case $n=0$. For the inductive step, suppose that $H^1_e(G_K, U_{n-1})$ is a closed subscheme of $H^1(G_K, U_{n-1})$, and let $H_{e,n} \subseteq H^1(G_K, U_n)$ be the preimage of $H^1_e(G_K, U_{n-1})$ under the map $H^1(G_K, U_n) \longrightarrow H^1(G_K, U_{n-1})$.

So $H_{e,n}$ is a closed subscheme of $H^1(G_K, U_n)$, containing $H^1_e(G_K, U_n)$ as a subfunctor.

Now since the central extension

$$1 \longrightarrow V_n \longrightarrow U_n \longrightarrow U_{n-1} \longrightarrow 1$$

is split as a sequence of varieties, it follows that

$$1 \longrightarrow V_{n, B_{\mathrm{cris}}^{\varphi=1}}(\Lambda) \longrightarrow U_{n, B_{\mathrm{cris}}^{\varphi=1}}(\Lambda) \longrightarrow U_{n-1, B_{\mathrm{cris}}^{\varphi=1}}(\Lambda) \longrightarrow 1$$

is topologically split and we have a long exact seq.

$$H^1(G_K, V_{n,B_{cris}^{\varphi=1}}) \xrightarrow{\alpha} H^1(G_K, U_{n,B_{cris}^{\varphi=1}}) \xrightarrow{\beta} H^1(G_K, U_{n-1,B_{cris}^{\varphi=1}}).$$

The kernel of the left-hand map α is the image of

$$H^0(G_K, U_{n-1,B_{cris}^{\varphi=1}}) = D_{cris}^{\varphi=1}(U_{n-1}) \text{ in } H^1(G_K, V_{n,B_{cris}^{\varphi=1}})$$

which is trivial since $D_{cris}^{\varphi=1}(U_{n-1}) = 1$. So $\ker(\alpha) = \{*\}$.

Now let $f : H^1(G_K, U_n) \to H^1(G_K, U_{n,B_{cris}^{\varphi=1}})$ be the natural map, and $f|_{H_{e,n}}$ its restriction to $H_{e,n}$. By construction, the composite $\beta \circ f|_{H_{e,n}}$ is zero, so $f|_{H_{e,n}}$ lifts to a map $\tilde{f} : H_{e,n} \to H^1(G_K, V_{n,B_{cris}^{\varphi=1}})$, (using that $H_{e,n}$ is representable). Since $\ker(\alpha) = \{*\}$, we have that

$$H^1_e(G_K, U_n) := \ker(f) = \ker(f|_{H_{e,n}}) = \ker(\tilde{f}).$$

But now $H^1(G_K, V_{n,B_{cris}^{\varphi=1}})$ is the functor

$$\Lambda \longmapsto H^1(G_K, B_{cris}^{\varphi=1} \otimes_{\mathbb{Q}_p} \Lambda \otimes_{\mathbb{Q}_p} V_n) = H^1(G_K, B_{cris}^{\varphi=1} \otimes_{\mathbb{Q}_p} V_n) \otimes_{\mathbb{Q}_p} \Lambda$$

which is clearly subrepresentable. So $\ker(\tilde{f})$ is the kernel of a morphism of pointed affine schemes, so is representable by a closed subscheme of $H^1(G_K, U_n)$. This completes the inductive step.

Thus, we now know that $H^1_e(G_K, U_n)$ is representable by a closed subscheme of $H^1(G_K, U_n)$ for all n, and in particlular we are done if the group U is unipotent. For the general case, we claim that $H^1(G_K, U)$ is the intersection of the preimages of $H^1_e(G_K, U_n) \subseteq H^1(G_K, U_n)$ for all n. This is an <u>exercise</u> in chasing limits, and left to the reader. $\square_e$

We've proved the theorem for $H^1_e(G_K, U)$, which is the main case we will be using in what follows. We outline the proofs in the remaining cases.

For $H^1_g(G_K, U)$, essentially the same proof works, except that we need a slightly different argument for why the map

$$H^1(G_K, V_{n,B_{dR}}) \longrightarrow H^1(G_K, U_{n,B_{dR}})$$

has trivial kernel. This kernel is given by the cokernel of

$$H^0(G_K, U_{n,B_{dR}}) \longrightarrow H^0(G_K, U_{n-1,B_{dR}}),$$ i.e. the cokernel of

$$\mathrm{Res}^K_{\mathbb{Q}_p} D_{dR}(U_n) \longrightarrow \mathrm{Res}^K_{\mathbb{Q}_p} D_{dR}(U_{n-1}). \qquad (*)$$

Since $\mathrm{Lie}(U_n)$ is a de Rham representation of G_K, the map ~~Res~~ $D_{dR}(\mathrm{Lie}(U_n)) \longrightarrow D_{dR}(\mathrm{Lie}(U_{n-1}))$ is surjective, and so $\circledast$ is also surjective. This implies that $H^1(G_K, V_{n,B_{dR}}) \longrightarrow H^1(G_K, U_{n,B_{dR}})$ has trivial kernel, and from here the proof of representability of $H^1_g(G_K, U)$ proceeds exactly as for $H^1_e(G_K, U)$.

For $H^1_f(G_K, U)$, we take a different tack, and prove

<u>Proposition</u>: Under the above assumptions,

$$H^1_f(G_K, U) = H^1_e(G_K, U).$$

<u>Lemma</u>: For any $\mathbb{Q}_p$-algebra Λ, the map

$$\begin{array}{ccc} D_{cris}(U)(\Lambda) & \longrightarrow & D_{cris}(U)(\Lambda) \\ w & \longmapsto & w^{-1}\varphi(w) \end{array}$$

is bijective. (Here, by mild abuse of notation, we write $D_{cris}(U)$ for $\mathrm{Res}^{K_0}_{\mathbb{Q}_p} D_{cris}(U)$.)

~~Proof of Lemma: We show by induction on n that the map~~ $\cancel{D_{cris}(U_n)(\Lambda) \longrightarrow D_{cris}(U_n)(\Lambda)}$ ~~is bijective for all n. The base case n=0 is trivial,~~

Proof of Lemma:

Let $W_{-n}D_{cris}(U) := D_{cris}(W_{-n}U)$, which is a subgroup-scheme of $D_{cris}(U)$, and let $D_{cris}(U)_n := D_{cris}(U)/W_{-n-1}D_{cris}(U)$. The crystalline Frobenius on $D_{cris}(U)$ restricts to a crystalline Frobenius on $W_{-n}D_{cris}(U)$ for all n, and so there is an induced crystalline Frobenius on $D_{cris}(U)_n$. We will show by induction that the map

$$D_{cris}(U)_n(\Lambda) \longrightarrow D_{cris}(U)_n(\Lambda) \qquad w \longmapsto w^{-1}\varphi(w)$$

is bijective for all n, from which the result follows by ~~induction.~~ taking an inverse limit.

The base case $n=0$ of the induction is trivial, so suppose inductively that $D_{cris}(U)_{n-1}(\Lambda) \longrightarrow D_{cris}(U)_{n-1}(\Lambda)$ is bijective. For $u \in D_{cris}(U)_n(\Lambda)$, let $\bar{u} \in D_{cris}(U)_{n-1}(\Lambda)$ denote its image, so that there is a unique $\bar{w} \in D_{cris}(U)_{n-1}(\Lambda)$ s.t. $\bar{u} = \bar{w}^{-1}\varphi(\bar{w})$. Let $w_0 \in D_{cris}(U)_n(\Lambda)$ be any lift, so that $w_0 u \varphi(w_0)^{-1} \in \ker(D_{cris}(U)_n(\Lambda) \to D_{cris}(U)_{n-1}(\Lambda))$

$$= \frac{D_{cris}(W_{-n}U)(\Lambda)}{D_{cris}(W_{-n-1}U)(\Lambda)} \subseteq D_{cris}(V_n \otimes \Lambda).$$
$$D_{cris}(V_n \otimes \Lambda) = D_{cris}(V_n) \otimes \Lambda$$

Since $D_{cris}^{\varphi=1}(V_n) = 0$ by our assumption on the weights, we know that the map $(1-\varphi) : \frac{D_{cris}(W_{-n}U)(\Lambda)}{D_{cris}(W_{-n-1}U)(\Lambda)} \longrightarrow \frac{D_{cris}(W_{-n}U)(\Lambda)}{D_{cris}(W_{-n-1}U)(\Lambda)}$

is an isomorphism of finitely generated Λ-modules.
So there exists a unique element $v \in \dfrac{D_{cris}(W_{-n}U)(\Lambda)}{D_{cris}(W_{-n-1}U)(\Lambda)}$
such that $w_0 u \varphi(w_0)^{-1} = v^{-1}\varphi(v)$.
It is now easy to see that $w := v w_0$ is the unique element of $D_{cris}(U)_n(\Lambda)$ s.t.

$$u = w^{-1}\varphi(w).$$

In other words, the map $D_{cris}(U)_n(\Lambda) \longrightarrow D_{cris}(U)_n(\Lambda)$ given by $w \longmapsto w^{-1}\varphi(w)$ is bijective, which is what we wanted to show. $\square$

~~Remark: There is a rather slicker proof of the same State lemma using our representability theorem. That is, if we let $\mathbb{Z}$ act on $\mathrm{Res}^{K_0}_{\mathbb{Q}_p}(D_{cris}(U))$ where the generator of $\mathbb{Z}$ acts on $\mathrm{Res}^{K_0}_{\mathbb{Q}_p}(D_{cris}(U))$ by the crystalline Frobenius φ, then we have~~

~~$H^0($~~

<u>Remark</u>: Here is a rather slicker conceptualisation of the same proof. Let $\mathbb{Z}$ act on $\mathrm{Res}^{K_0}_{\mathbb{Q}_p}(D_{cris}(U))$, where $n \in \mathbb{Z}$ acts by φ^n. Equipping $\mathrm{Res}^{K_0}_{\mathbb{Q}_p}(D_{cris}(U))$ with the weight filtration induced from the weight filtration on U, we have

that $H^0(\mathbb{Z}, \mathrm{gr}^W_{-n}(\mathrm{Res}^{k_0}_{\mathbb{Q}_p} D_{\mathrm{cris}}(U))) = 0$ for all n,
since $\mathrm{gr}^W_{-n} \mathrm{Res}^{k_0}_{\mathbb{Q}_p} D_{\mathrm{cris}}(U) \leq \cancel{\bigoplus D_{\mathrm{cris}}} \mathrm{Res}^{k_0}_{\mathbb{Q}_p} D_{\mathrm{cris}}(V_n$
and the latter has no φ-invariants by our assumptio
on the weights. So our representability theorem* implies that

$$H^1(\mathbb{Z}, \mathrm{Res}^{k_0}_{\mathbb{Q}_p} D_{\mathrm{cris}}(U)) \text{ is a closed subscheme}$$

$$\text{of } \prod_n H^1(\mathbb{Z}, \mathrm{gr}^W_{-n}(\mathrm{Res}^{k_0}_{\mathbb{Q}_p} D_{\mathrm{cris}}(U))) = 0,$$

using that $\dim H^1(\mathbb{Z}, \mathrm{gr}^W_{-n}(\mathrm{Res}^{k_0}_{\mathbb{Q}_p} D_{\mathrm{cris}}(U)))$

$$= \dim H^0(\mathbb{Z}, \mathrm{gr}^W_{-n}(\mathrm{Res}^{k_0}_{\mathbb{Q}_p} D_{\mathrm{cris}}(U))) = 0$$

by the above. So

$$H^1(\mathbb{Z}, \mathrm{Res}^{k_0}_{\mathbb{Q}_p} D_{\mathrm{cris}}(U)) = 1 \quad \text{and} \quad H^0(\mathbb{Z}, \mathrm{Res}^{k_0}_{\mathbb{Q}_p} D_{\mathrm{cris}}(U)) =$$

which says exactly that the map $w \longmapsto w^{-1}\varphi(w)$ is bijective.

*Strictly speaking, we only stated our representability theorem for G profinite. It still holds for $G = \mathbb{Z}$, by the same proof.

Proof of Proposition

Suppose that $\xi \in Z^1(G_K, U(\Lambda))$ is a 1-cocycle whose class lies in $H^1_f(G_K, U(\Lambda))$. Explicitly, this means that there is some $u \in U(B_{cris} \otimes_{\mathbb{Q}_p} \Lambda)$ such that

$$\xi(\sigma) = u^{-1}\sigma(u)$$

for all $\sigma \in G_K$. Since $\xi(\sigma) \in U(\Lambda)$, it is invariant under the action of φ, which implies that

$$\varphi(u)^{-1}\varphi\sigma(u) = u^{-1}\sigma(u)$$

for all $\sigma \in G_K$. Rearranging and using that φ commutes with the action of G_K gives that

$$\sigma(u \cdot \varphi(u)^{-1}) = u \cdot \varphi(u)^{-1}$$

so $u \cdot \varphi(u)^{-1} \in U(B_{cris} \otimes_{\mathbb{Q}_p} \Lambda)^{G_K} = D_{cris}(U)(\Lambda)$.

By the preceding lemma, there exists a unique $w \in D_{cris}(U)(\Lambda)$ with

$$w^{-1}\varphi(w) = u \cdot \varphi(u)^{-1}.$$

This implies that $\varphi(wu) = wu$, so $wu \in U(B_{cris}^{\varphi=1} \otimes_{\mathbb{Q}_p} \Lambda)$, and since $w \in U(B_{cris} \otimes \Lambda)^{G_K}$ is G_K-fixed we also have

$$(wu)^{-1} \cdot \sigma(wu) = u^{-1}\sigma(u) = \xi(\sigma)$$

for all $\sigma \in G_K$. This shows that $[\xi] \in H^1_e(G_K, U(\Lambda))$, as desired. $\square$

The non-abelian Bloch–Kato exponential

The local Bloch–Kato Selmer scheme $H^1_e(G_K, U)$ can be understood quite explicitly thanks to a non-abelian version of the Bloch–Kato exponential. We temporarily relax our assumption on the weights of U (but keep our assumptions on finite generation, de Rhamness etc.)

Theorem: (non-abelian Bloch–Kato exponential)
There is a canonical isomorphism

$$\mathrm{Res}^K_{\mathbb{Q}_p} D^+_{\mathrm{dR}}(U) \backslash \mathrm{Res}^K_{\mathbb{Q}_p} D_{\mathrm{dR}}(U) / D^{\varphi=1}_{\mathrm{cris}}(U) \xrightarrow{\sim} H^1_e(G_K, U)$$

of functors $\underline{\mathrm{Alg}}_{\mathbb{Q}_p} \longrightarrow \underline{\mathrm{Set}}_*$, called the <u>Bloch–Kato exponential</u> $\exp_{BK}$. Its inverse is called the <u>Bloch–Kato logarithm</u> $\log_{BK}$.

<u>Application</u>: Suppose we're back in the case that all the weights of U are negative. This implies that $D_{cris}^{\varphi=1}(U)=1$, and so the Bloch–Kato exponential gives us a completely explicit description of the local Bloch–Kato Selmer scheme, as

$$H^1_e(G_K,U) = H^1_f(G_K,U) \cong \mathrm{Res}^K_{\mathbb{Q}_p} D_{dR}(U) / \mathrm{Res}^K_{\mathbb{Q}_p} D^+_{dR}(U).$$

As we shall see in the fifth Problems Sheet, the right-hand side is an affine space of dimension

$$[K:\mathbb{Q}_p]\cdot(\dim_K D_{dR}(U) - \dim_K D^+_{dR}(U)).$$

$$= [K:\mathbb{Q}_p]\cdot(\dim_K D_{dR}(\mathrm{Lie}(U)) - \dim_K D^+_{dR}(\mathrm{Lie}(U))).$$

<u>Corollary</u>: in the above setting, the maps

$$H^1_f(G_K,U_n) \longrightarrow H^1_f(G_K,U_{n-1}) \quad (*)$$

are surjective for all n, and the action of $H^1(G_v,V_n)$ on $H^1_f(G_K,U_n)$ restricts to an action of $H^1_f(G_K,V_n)$ on $H^1_f(G_K,U_n)$ which is simply transitive on the fibres of $(*)$.

In other words, $H^1_f(G_K,U_n)$ is a ~~torsor~~ $H^1_f(G_K,V_n)$-torsor over $H^1_f(G_K,U_{n-1})$.

Proof of Corollary:

The central extension

$$1 \longrightarrow V_n \longrightarrow U_n \longrightarrow U_{n-1} \longrightarrow 1$$

induces central extensions

$$1 \longrightarrow D_{dR}(V_n) \longrightarrow D_{dR}(U_n) \longrightarrow D_{dR}(U_{n-1}) \longrightarrow 1$$

$$1 \longrightarrow D_{dR}^{+}(V_n) \longrightarrow D_{dR}^{+}(U_n) \longrightarrow D_{dR}^{+}(U_{n-1}) \longrightarrow 1$$

of unipotent groups over K (using that D_{dR} and D_{dR}^{+} are exact functors on de Rham representations). Using this, it is easy to verify that $\operatorname{Res}^K_{\mathbb{Q}_p} D_{dR}^{+}(U_n) \backslash \operatorname{Res}^K_{\mathbb{Q}_p} D_{dR}(U_n)$ is a $\operatorname{Res}^K_{\mathbb{Q}_p} D_{dR}^{+}(V_n) \backslash \operatorname{Res}^K_{\mathbb{Q}_p} D_{dR}(V_n)$-torsor over $\operatorname{Res}^K_{\mathbb{Q}_p} D_{dR}^{+}(U_{n-1}) \backslash \operatorname{Res}^K_{\mathbb{Q}_p} D_{dR}(U_{n-1})$ (with respect to the action ~~of~~ by multiplication). Naturality of the Bloch–Kato exponential implies then that $H^1_f(G_K, U_n)$ is a $H^1_f(G_K, V_n)$-torsor over $H^1_f(G_K, U_{n-1})$ as claimed.

$\square$

Proof of the non-abelian Bloch–Kato exponential

To construct the exponential map, we need a preparatory lemma.

Lemma: Let $U/\mathbb{Q}_p$ be a ~~finitely generated~~ pro-unipotent group. Then for all $\mathbb{Q}_p$-algebras Λ

a) the multiplication map

$$U(B_{dR}^+ \otimes \Lambda) \times U(B_{cris}^{\varphi=1} \otimes \Lambda) \longrightarrow U(B_{dR} \otimes \Lambda)$$

is surjective; and

b) $U(B_{dR}^+ \otimes \Lambda) \cap U(B_{cris}^{\varphi=1} \otimes \Lambda) = U(\Lambda)$

compatibly with topologies.

Proof: The validity of (b) is preserved under small limits, so it suffices to prove it when U is unipotent. Identifying U with its Lie algebra $\mathrm{Lie}(U)$, we have

$$\begin{aligned} U(B_{dR}^+ \otimes \Lambda) \cap U(B_{cris}^{\varphi=1} \otimes \Lambda) &= (\mathrm{Lie}(U) \otimes B_{dR}^+ \otimes \Lambda) \cap (\mathrm{Lie}(U) \otimes B_{cris}^{\varphi=1} \otimes \Lambda) \\ &= \mathrm{Lie}(U) \otimes \Lambda = U(\Lambda) \end{aligned}$$

using that $B_{dR}^+ \cap B_{cris}^{\varphi=1} = \mathbb{Q}_p$ in the second equality. (From the exact sequence

$$0 \to \mathbb{Q}_p \to B_{dR}^+ \oplus B_{cris}^{\varphi=1} \longrightarrow B_{dR} \to 0.)$$

So we have proved (b).

The validity of (a) is not a priori preserved under small limits, so we instead prove a stronger statement which is. Fix a complementary subspace $\overline{B}_{cris}^{\varphi=1} \subset B_{cris}^{\varphi=1}$ to $\mathbb{Q}_p \subset B_{cris}^{\varphi=1}$, so that $B_{cris}^{\varphi=1} = \mathbb{Q}_p \oplus \overline{B}_{cris}^{\varphi=1}$. Note that $\overline{B}_{cris}^{\varphi=1}$ is not a ring. Nonetheless, for any unipotent group $U/\mathbb{Q}_p$, we can define $U(\overline{B}_{cris}^{\varphi=1} \otimes \Lambda)$ to be the subset of $U(B_{cris}^{\varphi=1} \otimes \Lambda)$ corresponding to

$$\mathrm{Lie}(U) \otimes \overline{B}_{cris}^{\varphi=1} \otimes \Lambda \subset \mathrm{Lie}(U) \otimes B_{cris}^{\varphi=1} \otimes \Lambda.$$

We extend this definition to pro-unipotent U by taking an inverse limit. We will prove

a') the multiplication map

$$U(B_{dR}^+ \otimes \Lambda) \times U(\overline{B}_{cris}^{\varphi=1} \otimes \Lambda) \to U(B_{dR} \otimes \Lambda) \qquad (*)$$

is bijective.

This trivially implies (a).

Now the validity of (a') is preserved under small limits, so ~~it~~ suffices to prove it when U is unipotent. When U is a vector group, $(*)$ is identified with the addition map

$$(\mathrm{Lie}(U) \otimes B_{dR}^+ \otimes \Lambda) \times (\mathrm{Lie}(U) \otimes \overline{B}_{cris}^{\varphi=1} \otimes \Lambda) \to (\mathrm{Lie}(U) \otimes B_{dR} \otimes \Lambda)$$

which is bijective since $B_{dR}^+ \oplus \overline{B}_{cris}^{\varphi=1} = B_{dR}$.

In general, we proceed inductively, writing U as a central extension

$$1 \longrightarrow V \longrightarrow U \longrightarrow U' \longrightarrow 1$$

where we know the result for U' and V (as V is abelian). ~~the map The group~~ For $B \in \{B_{dR}^+, \bar{B}_{cris}^{\varphi=1}, B_{dR}\}$, the group $V(B\otimes\Lambda)$ acts via multiplication on $U(B\otimes\Lambda)$, ~~and the quotient is~~ making $U(B\otimes\Lambda)$ into a $V(B\otimes\Lambda)$-torsor over $U'(B\otimes\Lambda)$. The map

$$U(B_{dR}^+\otimes\Lambda) \times U(\bar{B}_{cris}^{\varphi=1}\otimes\Lambda) \longrightarrow U(B_{dR}\otimes\Lambda)$$

is a morphism of torsors under

$$V(B_{dR}^+\otimes\Lambda) \times V(\bar{B}_{cris}^{\varphi=1}\otimes\Lambda) \cong V(B_{dR}\otimes\Lambda)$$

over

$$U'(B_{dR}^+\otimes\Lambda) \times U'(\bar{B}_{cris}^{\varphi=1}\otimes\Lambda) \cong U'(B_{dR}\otimes\Lambda)$$

and hence is an isomorphism $\square$

The Bloch–Kato exponential

$$\frac{\mathrm{Res}^K_{\mathbb{Q}_p} D_{dR}(U)}{\mathrm{Res}^K_{\mathbb{Q}_p} D^+_{dR}(U) \quad D^{\varphi=1}_{cris}(U)} \longrightarrow H^1_e(G_K, U)$$

is defined as follows. Let Λ be a $\mathbb{Q}_p$-algebra and

$$u \in \mathrm{Res}^K_{\mathbb{Q}_p} D_{dR}(U)(\Lambda) = U(B_{dR} \underset{\mathbb{Q}_p}{\otimes} \Lambda)^{G_K}.$$

We know by the lemma that u can be written as $u_{dR} u_{cris}^{-1}$ for $u_{dR} \in U(B^+_{dR} \otimes \Lambda)$ and $u_{cris} \in U(B^{\varphi=1}_{cris} \otimes \Lambda)$.

Since $u = u_{dR} u_{cris}^{-1}$ is G_K-fixed, we have

$$u_{dR}^{-1} \sigma(u_{dR}) = u_{cris}^{-1} \sigma(u_{cris})$$

for all $\sigma \in G_K$. So

$$u_{dR}^{-1} \sigma(u_{dR}) = u_{cris}^{-1} \sigma(u_{cris}) \in U(B^+_{dR} \otimes \Lambda) \cap U(B^{\varphi=1}_{cris} \otimes \Lambda) = U(\Lambda)$$

and so the map

$$\begin{aligned} \xi : G_K &\longrightarrow U(\Lambda) \\ \sigma &\longmapsto u_{cris}^{-1} \sigma(u_{cris}) \end{aligned}$$

is a continuous cocycle. The class of ξ lies in $H^1_e(G_K, U(\Lambda))$ since it is the coboundary of $u_{cris} \in U(B^{\varphi=1}_{cris} \otimes \Lambda)$.

<u>Lemma:</u> the above construction defines a map

$$\frac{\mathrm{Res}^K_{\mathbb{Q}_p} D_{dR}(U)}{\mathrm{Res}^K_{\mathbb{Q}_p} D^+_{dR}(U) \quad D^{\varphi=1}_{cris}(U)} \longrightarrow H^1_e(G_K, U)$$

of functors.

<u>Proof</u>: Suppose that $u = u_{dR} u_{cris}^{-1}$ and $u' = u'_{dR} u'^{-1}_{cris}$ represent the same element of

$$\begin{tikzcd}[column sep=small] & U(B_{dR}\otimes\Lambda)^{G_K} \arrow[dl, dash] \arrow[dr, dash] & \\ U(B^+_{dR}\otimes\Lambda)^{G_K} & & U(B_{cris}^{\varphi=1}\otimes\Lambda)^{G_K}. \end{tikzcd}$$

Let ξ and ξ' be the corresponding cocycles

So $u' = w_{dR} u w_{cris}^{-1}$ for $w_{dR} \in U(B^+_{dR}\otimes\Lambda)^{G_K}$
$w_{cris} \in U(B_{cris}^{\varphi=1}\otimes\Lambda)^{G_K}$

Rearranging,

$$w^{-1} := u'^{-1}_{dR} w_{dR} u_{dR} = u'^{-1}_{cris} w_{cris} u_{cris} \in U(B^+_{dR}\otimes\Lambda) \cap U(B_{cris}^{\varphi=1}\otimes\Lambda)$$
$$= U(\Lambda)$$

So $u'_{dR} = w_{dR} u_{dR} w$
and $u'_{cris} = w_{cris} u_{cris} w$

Now for $\sigma \in G_K$, we have

$$u'^{-1}_{cris} \cdot \sigma(u'_{cris}) = w^{-1} \cdot u^{-1}_{cris} \underbrace{w^{-1}_{cris}\sigma(w_{cris})}_{1} \sigma(u_{cris}) \sigma(w)$$

~~so $\sigma \mapsto u'^{-1}_{cris}\sigma(u'_{cris})$ and $\sigma \mapsto u^{-1}_{cris}\sigma(u_{cris})$ represent the same class in $H^1(G_K, U(\Lambda))$. □~~

i.e. $\xi' = \xi^w$ is the w-twist of ξ.

So $[\xi'] = [\xi]$ in $H^1_e(G_K, U(\Lambda))$. □

Next, we need to construct the inverse map, the Bloch-Kato logarithm

$$H^1_e(G_K, U) \longrightarrow \mathrm{Res}^K_{\mathbb{Q}_p} D^+_{dR}(U) \backslash \mathrm{Res}^K_{\mathbb{Q}_p} D_{dR}(U) / D_{cris}^{\varphi=1}(U)$$

<u>Lemma</u>: Let U be a de Rham representation of G_K on a finitely generated pro-unipotent group. Then for all $\mathbb{Q}_p$-algebras Λ, the map

$$H^1(G_K, U(B^+_{dR} \otimes \Lambda)) \longrightarrow H^1(G_K, U(B_{dR} \otimes \Lambda))$$

has trivial kernel.

<u>Proof</u>: If V is a de Rham representation of G_K, then a result of Bloch and Kato implies that

$$H^1(G_K, B^+_{dR} \otimes V) \longrightarrow H^1(G_K, B_{dR} \otimes V)$$

is injective. This implies the lemma for U a vector group. The case of U unipotent follows by an induction using the long exact sequence in cohomology, and the general case follows by taking an inverse limit. $\square$

Hence if

$$[\xi] \in H^1_e(G_K, U(\Lambda)) = \ker\left(H^1(G_K, U(\Lambda)) \to H^1(G_K, U(B_{cris}^{\varphi=1} \otimes \Lambda))\right)$$

then $[\xi]$ maps to zero in $H^1(G_K, U(B_{dR} \otimes \Lambda))$, and so by the lemma $[\xi]$ maps to zero in $H^1(G_K, U(B_{dR}^+ \otimes \Lambda))$ also. Choosing a representing cocycle $\xi : G_K \to U(\Lambda)$, we know that ξ is the coboundary of an element $u_{cris} \in U(B_{cris}^{\varphi=1} \otimes \Lambda)$, and also the coboundary of some $u_{dR} \in U(B_{dR}^+ \otimes \Lambda)$. So

$$\xi(\sigma) = u_{cris}^{-1} \sigma(u_{cris}) = u_{dR}^{-1} \sigma(u_{dR})$$

so $u := u_{dR} u_{cris}^{-1}$ is G_K-fixed, i.e. $u \in U(B_{dR} \otimes \Lambda)^{G_K}$.

<u>Lemma</u>: the map

$$H^1_e(G_K, U) \longrightarrow \mathrm{Res}^K_{\mathbb{Q}_p} D_{dR}^+(U) \backslash \mathrm{Res}^K_{\mathbb{Q}_p} D_{dR}(U) / D_{cris}^{\varphi=1}(U)$$

$$[\xi] \longmapsto [u]$$

is well-defined.

<u>Proof</u>: omitted.

The above map is called the <u>Bloch–Kato logarithm</u>.

It is easy to see that this is inverse to the Bloch–Kato exponential, so both are isomorphisms of functors. $\square$

Global Selmer schemes

Let now K be a number field, and let $U/\mathbb{Q}_p$ be a finitely generated pro-unipotent group endowed with a continuous action of G_K. Assume moreover:

- the action of G_K on U is ramified at only finitely many places;
- U is de Rham at all places $v|p$ of K; and
- U is endowed with a separated G_K-stable weight filtration

  $$U = W_{-1}U \trianglerighteq W_{-2}U \trianglerighteq W_{-3}U \trianglerighteq \dots$$

  whose graded pieces are pure of the corresponding weights at all finite places of K.

We want to define a "<u>global Selmer scheme</u>" inside $H^1(G_K, U)$, cut out by local conditions at each finite place of U. As in the abelian case, it is convenient to allow ourselves some flexibility in exactly which local conditions we impose.

<u>Definition</u> A <u>Selmer structure $\mathfrak{S}$</u> for U is a choice of closed subscheme $\mathfrak{S}_v \subseteq H^1(G_v, U)$ for each finite place v of K, such that $\mathfrak{S}_v = H^1_{nr}(G_v, U)$ for all but finitely many $v \nmid p$. (N.B. $H^1_{nr}(G_v, U) = \{*\}$ for ~~almost all v~~ all $v \nmid p$.)

Definition: Given a Selmer structure $\mathfrak{S}$ on U, we define the associated Selmer scheme

$$\mathrm{Sel}_{\mathfrak{S},U} \subseteq H^1(G_K, U)$$

to be the preimage of $\prod_{v\nmid\infty} \mathfrak{S}_v \subseteq \prod_{v\nmid\infty} H^1(G_v, U)$ under the product of the localisation maps

$$H^1(G_K, U) \longrightarrow \prod_{v\nmid\infty} H^1(G_v, U)$$

Example: Suppose that S is a finite set of finite places of K. Define a Selmer structure $\mathfrak{S}$ on U by

$$\mathfrak{S}_v = \begin{cases} H^1_{nr}(G_v, U) & \text{if } v\nmid p,\ v\notin S \\ H^1(G_v, U) & \text{if } v\nmid p,\ v\in S \\ H^1_f(G_v, U) & \text{if } v\mid p,\ v\notin S \\ H^1_g(G_v, U) & \text{if } v\mid p,\ v\in S. \end{cases}$$

The associated global Selmer scheme is denoted

$$H^1_{f,S}(G_K, U)$$

and parametrises cohomology classes which are unramified outside $S\cup\{v\mid p\}$, crystalline at $\{v\mid p\}\setminus S$ and de Rham at $S\cap\{v\mid p\}$.

There is something to be proved in the above definition, namely that the functor $\mathrm{Sel}_{\mathfrak{S},U}$ is indeed representable by an affine $\mathbb{Q}_p$-scheme. Note that since G_K does not satisfy property (F), this is not immediate.

Proposition: $\mathrm{Sel}_{\mathfrak{S},U}$ is representable by an affine $\mathbb{Q}_p$-scheme.

Lemma: Let T be a finite set of finite places of K, containing

- all places where the action of G_K on U is ramified
- all places dividing p
- all places v for which $\mathfrak{S}_v \neq \{*\}$.

(Such a finite set T exists by assumption.) Then

$$\mathrm{Sel}_{\mathfrak{S},U} \subseteq H^1(G_{K,T}, U)$$

largest quotient of G_K unramified outside T.

as subfunctors of $H^1(G_K, U)$.

Proof: Consider the non-abelian inflation-restriction exact sequence

$$1 \longrightarrow H^1(G_{K,T}, U) \hookrightarrow H^1(G_K, U) \longrightarrow \underset{\displaystyle \mathrm{Hom}^{\mathrm{out}}(N,U)}{\overset{\displaystyle H^1(N,U)}{\|}}$$

where $N = \ker(G_K \twoheadrightarrow G_{K,T})$ is the closed normal subgroup generated topologically by the conjugates of the inertia groups I_v for $v \notin T$.

Now let $\xi \in Z^1(G_K, U(\Lambda))$ be a cocycle representing an element of $\mathrm{Sel}_{\mathfrak{S},U}(\Lambda) \subseteq H^1(G_K, U(\Lambda))$

We know that

$[\xi|_{I_v}] \in \mathfrak{S}_v(1) = \{*\}$

inside $H^1(I_v, U(1)) = \mathrm{Hom}^{cont}(I_v, U(1))$

for all $v \notin T$, so $\xi|_{I_v} = 1$. But the cocycle condition implies that $\ker(\xi) := \{\sigma \in G_K : \xi(\sigma) = 1\}$ is a closed normal subgroup of G_K. So we deduce that $N \subseteq \ker(\xi)$, i.e. $\xi|_N = 1$. By non-abelian inflation-restriction, this implies that $[\xi] \in H^1(G_{K,T}, U(1))$ as claimed.

✓lemma

Now to finish the proof of the proposition, we observe that the group $G_{K,T}$ does have property (F) by Hasse–Minkowski, so $H^1(G_{K,T}, U)$ is representable by our representability theorem. The global Selmer scheme $\mathrm{Sel}_{\mathfrak{S},U}$ is then the preimage of $\prod_{v \nmid \infty} \mathfrak{S}_v \subseteq \prod_{v \nmid \infty} H^1(G_v, U)$ under the localisation map $H^1(G_{K,T}, U) \longrightarrow \prod_{v \nmid \infty} H^1(G_v, U)$, so is representable by a closed subscheme of $H^1(G_{K,T}, U)$.

$\square$

The proof of the proposition shows that $Sel_{\mathfrak{S},U}$ is non-canonically a closed subscheme of

$$\prod_n H^1(G_{K,T}, V_n).$$

In particular, when U is unipotent, one has the dimension inequality

$$\dim_{\mathbb{Q}_p} Sel_{\mathfrak{S},U} \leq \sum_n \dim_{\mathbb{Q}_p} H^1(G_{K,T}, V_n).$$

This inequality, however, depends on the set T, and hence on the Selmer structure $\mathfrak{S}$, in a rather opaque way. We will want to have access to a rather better bound in the cases we care about.

<u>Proposition</u>: Suppose that the Selmer structure $\mathfrak{S}$ has $\mathfrak{S}_v = H^1_f(G_v, U)$ for all $v \mid p$. Then $Sel_{\mathfrak{S},U}$ is a closed subscheme of

$$\prod_{v \nmid p} \mathfrak{S}_v \times \prod_n H^1_f(G_K, V_n).$$

In particular, if U is unipotent, then

$$\dim_{\mathbb{Q}_p} Sel_{\mathfrak{S},U} \leq \sum_{v \nmid p} \dim_{\mathbb{Q}_p} \mathfrak{S}_v + \sum_n \dim_{\mathbb{Q}_p} H^1_f(G_K, V_n).$$

For the proof, we adopt some auxiliary notation. Let T be as in the preceding lemma, and define $\mathrm{Sel}_{\mathfrak{S},n}$ to be the pullback

$$\begin{tikzcd}
\mathrm{Sel}_{\mathfrak{S},n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & H^1(G_{K,T}, U_n) \arrow[d] \\
\prod_{v|p} H^1_f(G_v, U_n) \times \prod_{v\in T_0} \mathfrak{S}_v \arrow[r] & \prod_{v\in T} H^1(G_v, U_n).
\end{tikzcd}$$

where $T_0 = T \setminus \{v|p\}$!

Explicitly, $\mathrm{Sel}_{\mathfrak{S},n}(\Lambda)$ is the set of tuples $(\xi, (\xi_v)_{v\in T_0})$ where

- $\xi \in H^1(G_{K,T}, U_n(\Lambda))$;
- $\xi_v \in \mathfrak{S}_v(\Lambda) \subseteq H^1(G_v, U(\Lambda))$ for all $v \in T_0$

such that

- $\xi|_{G_v} \in H^1_f(G_{K,T}, U_n(\Lambda))$ for all $v|p$; and
- $\xi|_{G_v}$ is equal to the image of ξ_v in $H^1(G_v, U_n(\Lambda))$ for all $v \in T_0$.

Each $\mathrm{Sel}_{\mathfrak{G},n}$ is representable by an affine $\mathbb{Q}_p$-scheme of finite type, the maps $U_n \to U_{n-1}$ induce projections $\mathrm{Sel}_{\mathfrak{G},n} \to \mathrm{Sel}_{\mathfrak{G},n-1}$ for all n, and we have $\mathrm{Sel}_{\mathfrak{G},U} = \varprojlim_n \mathrm{Sel}_{\mathfrak{G},n}$. So the proposition follows from (using that $\mathrm{Sel}_{\mathfrak{G},0} = \prod_{v \in T_0} \mathfrak{G}_v$)

<u>Lemma</u>: For all $n \geq 1$, the pointwise image of $\mathrm{Sel}_{\mathfrak{G},n} \longrightarrow \mathrm{Sel}_{\mathfrak{G},n-1}$ is representable by a closed subscheme of $\mathrm{Sel}_{\mathfrak{G},n-1}$, and $\mathrm{Sel}_{\mathfrak{G},n}$ has the structure of a $H^1_f(G_K, V_n)$-torsor over this image.

<u>Proof</u>: Recall from the proof of the representability theorem that there is a natural right action of $H^1(G_{K,T}, V_n)$ on $H^1(G_{K,T}, U_n)$ ~~which acts~~ by pointwise multiplication of cocycles, and that this acts simply transitively on the fibres of $H^1(G_{K,T}, U_n) \to H^1(G_{K,T}, U_{n-1})$. This restricts to an action of $H^1_f(G_K, V_n) \subseteq H^1(G_{K,T}, V_n)$ on $\mathrm{Sel}_{\mathfrak{G},n}$, given by $\alpha : (\xi, (\xi_v)_{v \in T_0}) \longmapsto (\xi \cdot \alpha, (\xi_v)_{v \in T_0})$ (Note that the assumption that $\alpha \in H^1(G_K, V_n)$ ensures that $(\xi \cdot \alpha)|_{G_v} = \xi|_{G_v}$ for $v \in T_0$ and that $(\xi \cdot \alpha)|_{G_v} \in H^1_f$ for $v | p$, so this action is well-defined on $\mathrm{Sel}_{\mathfrak{G},U}$.

So there are two things to be proved.

<u>Claim 1</u>: The above action of $H^1_f(G_K, V_n)$ is simply transitive on the fibres of $\mathrm{Sel}_{\mathfrak{G},n} \longrightarrow \mathrm{Sel}_{\mathfrak{G},n-1}$.

<u>Claim 2</u>: The image of $\mathrm{Sel}_{\mathfrak{G},n} \longrightarrow \mathrm{Sel}_{\mathfrak{G},n-1}$ is a closed subscheme of $\mathrm{Sel}_{\mathfrak{G},n-1}$.

<u>Proof of claim 1</u>: It is clear that $H^1_f(G_K, V_n)$ acts freely, the point is that it acts ~~without fixed p~~ transitively. For this, let $(\xi_1, (\xi_v)_{v\in T_0})$ and $(\xi_2, (\xi_v)_{v\in T_0})$ be two points in the same fibre. We know that $\xi_2 = \xi_1 \cdot \alpha$ for some $\alpha \in H^1(G_{K,T}, V_n)$. ~~and that~~ Since the image of ξ_v in $H^1(G_v, U_n)$ is equal to both $\xi_1|_{G_v}$ and $\xi_2|_{G_v}$ for $v \in T_0$, we deduce that $\alpha|_{G_v} = 1$ (since the action of $H^1(G_v, V_n)$ on $H^1(G_v, U_n)$ is ~~fail~~ free). Similarly, ~~we dedu~~ since $H^1_f(G_v, V_n)$ acts simply transitively on the fibres of $H^1_f(G_v, U_n) \longrightarrow H^1_f(G_v, U_{n-1})$ for $v \mid p$ and both $\xi_1|_{G_v}$ and $\xi_2|_{G_v} = \xi_1 \cdot \alpha|_{G_v}$ lie in H^1_f, we deduce that $\alpha|_{G_v} \in H^1_f(G_v, V_n)$. It follows that $\alpha \in H^1_f(G_K, V_n)$ proving Claim 1.

For the proof of Claim 2, let $Sel'_{\tilde{G},n-1}$ denote the image of $Sel_{\tilde{G},n} \to Sel^{\#}_{\tilde{G},n-1}$, and let $Sel''_{\tilde{G},n-1} \subseteq Sel_{\tilde{G},n-1}$ denote the subfunctor consisting of tuples $(\xi, (\xi_v)_{v \in T_0})$ for which ξ lies in the image of $H^1(G_{K,T}, U_n) \to H^1(G_{K,T}, U_{n-1})$. So we have

$$Sel'_{\tilde{G},n-1} \subseteq Sel''_{\tilde{G},n-1} \subseteq Sel_{\tilde{G},n-1}.$$

<u>Claim 2a</u>: $Sel''_{\tilde{G},n-1} \subseteq Sel_{\tilde{G},n-1}$ is representable by a closed subscheme.

<u>Proof</u>: We know from the proof of the representability theorem that the image of $H^1(G_{K,T}, U_n) \to H^1(G_{K,T}, U_{n-1})$ is representable by a closed subscheme of the latter. $Sel''_{\tilde{G},n-1}$ is the inverse image of this closed subscheme by the projection $Sel_{\tilde{G},n-1} \to H^1(G_{K,T}, U_{n-1})$. $\checkmark_{\text{claim 2a}}$.

<u>Claim 2b</u>: $Sel'_{\tilde{G},n-1} \subseteq Sel''_{\tilde{G},n-1}$ is representable by a closed subscheme.

Let us write

$$A_n := \prod_{v|p} \frac{H^1(G_v, V_n)}{H^1_f(G_v, V_n)} \times \prod_{v \in T_0} H^1(G_v, V_n)$$

and let C_n denote the cokernel of the natural map $H^1(G_{K,T}, V_n) \longrightarrow A_n$, so A_n and C_n are vector groups. For any $(\xi, (\xi_v)_{v\in T_0})$ in $\mathrm{Sel}''_{G,n-1}(\Lambda)$, we know that ξ lifts (by definition) to some $\tilde{\xi} \in H^1(G_{K,T}, U_n(\Lambda))$. For $v \in T_0$, we know that $\tilde{\xi}|_{G_v}$ and the image of ξ_v in $H^1(G_v, U_n(\Lambda))$ lie in the same fibre of $H^1(G_v, U_n(\Lambda)) \to H^1(G_v, U_{n-1}(\Lambda))$, so differ by the action of some $\alpha_v \in H^1(G_v, V_n(\Lambda))$. Similarly, for $v|p$, we know $\tilde{\xi}|_{G_v}$ lies over a point in $H^1_f(G_v, U_{n-1}(\Lambda))$, so there ~~difference~~ is some $\alpha_v \in H^1(G_v, V_n(\Lambda))$ such that $\tilde{\xi}|_{G_v} \cdot \alpha_v \in H^1_f(G_v, U_n(\Lambda))$. The point $(\alpha_v)_{v\in T}$ defines a Λ-point of A_n, and it is easy to see that the image of this point in C_n is independent of the choice of lift $\tilde{\xi}$. So we have defined a map $\delta' : \mathrm{Sel}''_{G,n-1} \longrightarrow C_n$. We claim that $\ker(\delta') = \mathrm{Sel}'_{G,n-1}$.

In one direction, if $(\xi, (\xi_v)_{v \in T_0})$ lies in $\mathrm{Sel}'_{\tilde{G}, n-1}$, then it is by definition the image of a point $(\tilde{\xi}, (\xi_v)_{v \in T_0})$ in $\mathrm{Sel}_{\tilde{G}, n}$. Taking this $\tilde{\xi}$ as the choice of lift, the definition of $\mathrm{Sel}_{\tilde{G}, n}$ implies that we have $\alpha_v = 1$ for all $v \in T_0$ and $\alpha_v \in H^1_f(G_v, V_n)$ for all $v \mid p$, so $(\xi, (\xi_v)_{v \in T_0})$ lies in the kernel of δ'.

Conversely, if $(\xi, (\xi_v)_{v \in T_0})$ lies in the kernel, then one may choose the lift $\tilde{\xi}$ such that $\alpha_v = 1$ for $v \in T_0$ and $\alpha_v \in H^1_f$ for $v \mid p$, so that $(\tilde{\xi}, (\xi_v)_{v \in T_0})$ lies in $\mathrm{Sel}_{\tilde{G}, n}$ and $(\xi, (\xi_v)_{v \in T_0}) \in \mathrm{Sel}'_{\tilde{G}, n-1}$ as claimed. This completes the proof of claim 2b, and hence of the proposition. $\square$