Lecture 9: Cohomology of pro-unipotent groups

Suppose that G is a profinite group acting continuously on a pro-unipotent group $U/\mathbb{Q}_p$ (meaning that $\mathrm{Lie}(U)$ is a cofiltered limit of finite-dimensional continuous representations of G). Then the action of G on the $\mathbb{Q}_p$-points $U(\mathbb{Q}_p)$ is continuous, so we have the non-abelian cohomology set

$$H^1(G, U(\mathbb{Q}_p)).$$

Under favourable circumstances, this cohomology set admits an algebraic structure, i.e. it is the $\mathbb{Q}_p$-points of an affine $\mathbb{Q}_p$-scheme. The resulting <u>cohomology scheme</u> is a first step towards defining Selmer schemes. that appear in the non-abelian Chabauty method.

We will construct this cohomology scheme by identifying its functor of points. For any $\mathbb{Q}_p$-algebra Λ, topologised in the usual way (by identifying $\Lambda \cong \mathbb{Q}_p^{\oplus I}$ for some set I), the action of G on $U(\Lambda)$ is again continuous, so we have the cohomology $H^i(G, U(\Lambda))$ for $i=0,1$ or $i\geq 2$ and U abelian.

The assignment $\Lambda \longmapsto H^1(G, U(\Lambda))$ is functorial, so defines a functor

$$H^1(G,U) : \mathrm{Alg}_{\mathbb{Q}_p} \longrightarrow \underline{\mathrm{Set}}_* = \{\text{pointed sets}\}.$$

The main theorem of this lecture gives a criterion for $H^1(G,U)$ to be representable by an affine $\mathbb{Q}_p$-scheme. For this, suppose that U is equipped with a separated ~~filtration~~ G-stable filtration

$$U = W_{-1}U \trianglerighteq W_{-2}U \trianglerighteq \dots$$

by normal subgroup-schemes of finite codimension in U such that $[W_{-i}U, W_{-j}U] \leq W_{-i-j}U$ for all $i,j \geq 1$. We adopt the notational conventions

$$U_n := U/W_{-n-1}U \quad \text{and} \quad V_n := W_{-n}U/W_{-n-1}.$$

V_n is a vector group, and we permit ourselves to conflate V_n with its associated vector space.

<u>Theorem</u>: Suppose that $H^0(G,V_n) = 0$ and $H^1(G,V_n)$ is finite-dimensional for all $n \geq 1$.

Then $H^1(G,U)$ is representable by an affine $\mathbb{Q}_p$-scheme, which is non-canonically isomorphic to a closed subscheme of $\prod_n H^1(G,V_n)$.

Remarks: 0. We denote the representing scheme also by $H^1(G,U)$ and call it the cohomology scheme.

1. In practice, we will apply this theorem (with one exception) in the case that U is finitely generated as a pro-unipotent group (e.g. U is unipotent). In this case, the condition that each $W_{-i}U$ is of finite codimension is automatic. (It's implied by the commutator condition.)

2. If U is unipotent, then the cohomology scheme $H^1(G,U)$ is of finite type, since it is a closed subscheme of the finite-dimensional affine space $\prod_n H^1(G,V_n)$. This gives the dimension bound

$$\dim_{\mathbb{Q}_p} H^1(G,U) \leq \sum_n \dim_{\mathbb{Q}_p} H^1(G,V_n).$$

3. We say that a profinite group G has property (F) just when for all $d \in \mathbb{N}$, G has only finitely many open subgroups of index d. If G has property (F), then the cohomology groups $H^i(G,V)$ are always finite-dimensional, for any i and any continuous representation V of G.

The base case

Our proof of the representability of $H^i(G, U)$ will be inductive: showing that each $H^i(G, U_n)$ is representable and taking a limit to obtain representability of $H^i(G, U)$. The base case $n=1$ amounts to dealing with the cohomology functors of vector groups, which amounts to the following computation.

Proposition: Let V be a continuous representation of G and let Λ be an $\mathbb{Q}_p$-algebra. Give $\Lambda \otimes V$ its natural topology from writing it as a direct sum of copies of $\mathbb{Q}_p$. Then the natural map

$$\Lambda \otimes_{\mathbb{Q}_p} H^i(G, V) \longrightarrow H^i(G, \Lambda \otimes_{\mathbb{Q}_p} V)$$

is an isomorphism.

Corollary: In the above setup, let $\mathbb{G}(V)$ be the vector group associated to V. If $H^i(G,V)$ is finite-dimensional, then the cohomology functor $H^i(G, \mathbb{G}(V))$ is representable by $\mathbb{G}(H^i(G,V))$. In general, $H^i(G, \mathbb{G}(V))$ is subrepresentable, i.e. is a subfunctor of a representable functor.

Proof of Corollary:

The proposition implies that $H^i(G, \mathbb{G}(V))$ is the functor

$$\Lambda \longmapsto \Lambda \otimes_{\mathbb{Q}_p} H^i(G,V).$$

If $H^i(G,V)$ is finite-dimensional, this is by definition the functor of points of the vector group $\mathbb{G}(H^i(G,V))$. In general, let $H^i(G,V)^*$ denote the dual of $H^i(G,V)$, viewed as a vector space ~~rather than a pro-finite-dim~~ (i.e. ignoring the pro-finite-dimensional vector space structure). The functor $H^i(G, \mathbb{G}(V))$ is a subfunctor of the functor

$$\Lambda \longmapsto \mathrm{Hom}_{\mathbb{Q}_p}(H^i(G,V)^*, \Lambda)$$

and this latter functor is representable by $\mathrm{Spec}(\mathrm{Sym}^\bullet(H^i(G,V)^*))$. $\square$

Proof of proposition:

If we write $\Lambda = \mathbb{Q}_p^{\oplus J}$ for a set J, then what we are wanting to prove is that the natural map

$$H^i(G,V)^{\oplus J} \longrightarrow H^i(G, V^{\oplus J})$$

is an isomorphism. We prove a slightly more general version of this for later use.

Let $(V_j)_{j\in J}$ be Hausdorff topological G-modules indexed by a set J. We claim that the map

$$\bigoplus_{j\in J} H^i(G,V_j) \longrightarrow H^i(G, \bigoplus_{j\in J} V_j)$$

is an isomorphism. For this, recall that $H^i(G,V_j)$ is the cohomology of a complex of abelian groups, whose ith term is the set $\mathrm{Map}(G^i, V_j)$ of continuous functions $G^i \longrightarrow V_j$. Since taking cohomology commutes with direct sums, it thus suffices to prove that the map

$$\bigoplus_{j\in J} \mathrm{Map}(G^i, V_j) \longrightarrow \mathrm{Map}(G^i, \bigoplus_{j\in J} V_j)$$

is an isomorphism of abelian groups. It is clearly injective; to show it is surjective we need to show that every continuous map $G^i \longrightarrow \bigoplus_{j\in J} V_j$ factors through $\bigoplus_{j\in J_0} V_j$ for some finite subset $J_0 \subseteq J$.

So suppose that $\xi : G^i \longrightarrow \bigoplus_{j\in J} V_j$ is a continuous function. Let ξ_j denote the j^{th} component of ξ, and let $J_1 = \{ j \in J : \xi_j \neq 0 \}$. For $j \in J_1$ choose an open neighbourhood $\mathcal{U}_j \subseteq V_j$ of 0 not containing $\mathrm{im}(\xi_j)$, and for J_0 a finite subset of J_1 let

$$\mathcal{U}_{J_0} = \bigoplus_{j\in J_0} V_j \oplus \bigoplus_{j \in J_1 \setminus J_0} \mathcal{U}_j \oplus \bigoplus_{j\in J\setminus J_1} V_j \subset \bigoplus_{j\in J} V_j .$$

These sets are open subsets of $\bigoplus_{j\in J} V_j$ and constitute an open covering (since every element of $\bigoplus_{j\in J} V_j$ has coordinate 0 at ~~almost~~ all but finitely many $j \in J$). Moreover, $\mathcal{U}_{J_0} \cup \mathcal{U}_{J_0'} \subseteq \mathcal{U}_{J_0 \cup J_0'}$. So by compactness of G^i we deduce that there exists a single J_0 such that $\mathrm{im}(\xi) \subseteq \mathcal{U}_{J_0}$. We must have $J_0 = J_1$, else there is some $j \in J_1 \setminus J_0$ such that $\mathrm{im}(\xi_j) \not\subseteq \mathcal{U}_j$ by construction. So $J_0 = J_1$ is finite and $\cancel{\xi}$ $\mathrm{im}(\xi) \subseteq \bigoplus_{j\in J_0} V_j$ as desired. $\square$.

<u>Remark</u>: It is not in general true that $\mathrm{Map}(G, -)$ commutes with filtered colimits when G is compact. A standard counterexample is to take $\mathbb{Z}_p$, and write it as the filtered colimit of its countable closed subsets. The direct limit topology agrees with the usual topology, but the identity map $\mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ doesn't factor through a countable subset

Inductive step:

Now let's return to the setup of the representability theorem. The previous discussion shows that each cohomology functor $H^1(G, V_n)$ is representable by a vector group. Since $U_1 = V_1$, this in particular shows that $H^1(G, U_1)$ is representable. We'll use this as the base case of an induction to show:

Lemma: $H^1(G, U_n)$ is representable for all $n \geq 1$.

For the proof, we suppose inductively that $H^1(G, U_{n-1})$ is representable by an affine $\mathbb{Q}_p$-scheme. Consider the central extension (G-equivariant)

$$1 \longrightarrow V_n \longrightarrow U_n \longrightarrow U_{n-1} \longrightarrow 1$$

of unipotent groups. We saw ~~earlier~~ in lecture 3 that the surjection $U_n \longrightarrow U_{n-1}$ is split as a morphism of $\mathbb{Q}_p$-schemes. This implies that the induced sequence

$$1 \longrightarrow V_n(\Lambda) \longrightarrow U_n(\Lambda) \longrightarrow U_{n-1}(\Lambda) \longrightarrow 1$$

is a G-equivariant topologically split central extension for all $\Lambda \in \mathrm{Alg}_{\mathbb{Q}_p}$.

Taking the long exact sequence in cohomology gives

$$1 \to H^0(G, V_n(\Lambda)) \to H^0(G, U_n(\Lambda)) \to H^0(G, U_{n-1}(\Lambda)) \to H^1(G, V_n(\Lambda)) \to H^1(G, U_n(\Lambda)) \to H^1(G, U_{n-1}(\Lambda)) \to H^2(G, V_n(\Lambda))$$

along with an action of $H^1(G, V_n(\Lambda))$ on $H^1(G, U_n(\Lambda))$. Since the whole construction is functorial in Λ, this gives us a long exact sequence

$$1 \to H^0(G, V_n) \to H^0(G, U_n) \to H^0(G, U_{n-1}) \xrightarrow{\delta^0} H^1(G, V_n) \to H^1(G, U_n) \to H^1(G, U_{n-1}) \xrightarrow{\delta^1} H^2(G, V_n)$$

of functors $\underline{\mathrm{Alg}}_{\mathbb{Q}_p} \to \underline{\mathrm{Set}}_*$, along with an action of $H^1(G, V_n)$ on $H^1(G, U_n)$.

Claim 1: $H^1(G, V_n)$ acts strictly transitively on the fibres of $H^1(G, U_n) \longrightarrow H^1(G, U_{n-1})$. That is, for all $\Lambda \in \underline{\mathrm{Alg}}_{\mathbb{Q}_p}$, $H^1(G, V_n(\Lambda))$ acts strictly transitively on the fibres of $H^1(G, U_n(\Lambda)) \to H^1(G, U_{n-1}(\Lambda))$.

Proof of claim 1: We know from the discussion of the action in the long exact sequence that $H^1(G, V_n(\Lambda))$ acts transitively on the fibres; the point here is that it acts freely. Let $[\xi] \in H^1(G, U_n(\Lambda))$ be the class of a 1-cocycle $\xi \in Z^1(G, U_n(\Lambda))$. We know that the stabiliser of $[\xi]$ under the action of $H^1(G, V_n(\Lambda))$ is the image of the coboundary map

$$\delta^0 : H^0(G, {}_\xi U_{n-1}(\Lambda)_\xi) \longrightarrow H^1(G, V_n(\Lambda))$$

from the ξ-twisted exact sequence

$$1 \to V_n(\Lambda) \longrightarrow {}_\xi U_n(\Lambda)_\xi \longrightarrow {}_\xi U_{n-1}(\Lambda)_\xi \longrightarrow 1.$$

~~But~~ For $m \leq n$, we have an exact sequence

$$1 \to V_m(\Lambda) \to {}_\xi U_m(\Lambda)_\xi \longrightarrow {}_\xi U_{m-1}(\Lambda)_\xi \to 1.$$

Since $H^0(G, V_m(\Lambda)) = \Lambda \otimes H^0(G, V_m) = 0$ by assumption, this shows inductively that $H^0(G, {}_\xi U_m(\Lambda)_\xi) = 1$ for all $m \leq n$. In particular, $\mathrm{Stab}([\xi]) = 1$ so $H^1(G, V_n(\Lambda))$ acts freely.

Claim 2: The kernel of $\delta^1 : H^1(G, U_{n-1}) \to H^2(G, V_n)$ is representable by an affine $\mathbb{Q}_p$-scheme.

Proof of claim 2: We know that $H^1(G, U_{n-1})$ is representable (by inductive assumption) and that $H^2(G, V_n)$ is subrepresentable. Embedding $H^2(G, V_n)$ as a subfunctor of a representable functor, we see that $\ker(\delta^1)$ is the kernel of a morphism of representable functors $\underline{Alg}_{\mathbb{Q}_p} \longrightarrow \underline{Set}_*$.

This implies that $\ker(\delta^1)$ itself is representable. ☑

(since the subcategory of representable functors is closed under small limits).

As a consequence of Claim 2, the map $H^1(G, U_n) \longrightarrow \ker(\delta^1)$ coming from the long exact sequence splits as a morphism of functors.

Indeed, we know that

$$H^1(G, U_n) \twoheadrightarrow \ker(\delta^1) \cong \mathrm{Spec}(R)$$

is a surjection of functors, so in particular the map

$$H^1(G, U_n)(R) \twoheadrightarrow \ker(\delta^1)(R) \cong \mathrm{Spec}(R)(R) = \mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(R, R)$$

is surjective.

Any element of $H^1(G,U_n)(R)$ lying over the identity in $\mathrm{Hom}_{\mathbb{Q}_p\text{-alg}}(R,R)$ determines via Yoneda a morphism $\mathrm{Spec}(R) \longrightarrow H^1(G,U_n)$ of functors, which is necessarily a splitting of $H^1(G,U_n) \twoheadrightarrow \ker(\delta^1) \cong \mathrm{Spec}(R)$.

So let $s : \ker(\delta^1) \longrightarrow H^1(G,U_n)$ be a splitting, and consider the map

$$\ker(\delta^1) \times H^1(G,V_n) \longrightarrow H^1(G,U_n) \qquad \circledast$$

of functors given by $(u,v) \longmapsto s(u)\cdot v$. Since both sides of $\circledast$ subject to $\ker(\delta^1)$ ~~and~~ with $H^1(G,V_n)$ acting transitively on the fibres, it follows that $\circledast$ is an isomorphism. So $H^1(G,U_n)$ is representable completing the inductive step.

Limit step:

Since the filtration on U is separated, we know that the map $U \longrightarrow \varprojlim_n U_n$ is an isomorphism.

So to complete the proof of the representability theorem, it suffices to show that the map

$$H^1(G, \varprojlim_n U_n) \longrightarrow \varprojlim_n H^1(G, U_n)$$

is an isomorphism of functors (since a small limit of representable functors is representable).

These kinds of things are typically elementary but tedious to verify. We give the complete argument this once.

Let Λ be a $\mathbb{Q}_p$-algebra. There are two things to show regarding the map

$$H^1(G, \varprojlim_n U_n(\Lambda)) \longrightarrow \varprojlim_n H^1(G, U_n(\Lambda)) \quad \circledast$$

Claim 1: $\circledast$ is injective

Proof: Suppose that $\xi, \xi' \in Z^1(G, \varprojlim_n U_n(\Lambda))$ are two cocycles whose images $\xi_n, \xi_n' \in Z^1(G, U_n(\Lambda))$ represent the same element of $H^1(G, U_n(\Lambda))$ for all n.

So there exists an element $u_n \in U_n(\Lambda)$ such that $\xi'_n = \xi_n^{u_n}$. The element u_n is unique: if u'_n were another then we would have

$$u_n'^{-1} \cdot \xi_n(\sigma) \cdot \sigma(u'_n) = \xi_n^{u'_n}(\sigma) = \xi_n^{u_n}(\sigma) = u_n^{-1} \cdot \xi_n(\sigma) \cdot \sigma(u_n)$$

for all $\sigma \in G$. In other words,

$$\xi_n(\sigma) \cdot \sigma(u'_n u_n^{-1}) \xi_n(\sigma)^{-1} = u'_n u_n^{-1}$$

so that $u'_n u_n^{-1}$ is fixed under the ξ_n-twisted G-action on $U_n(\Lambda)$. But we saw earlier that $H^0(G, {}_{\xi_n}U_n(\Lambda)) = 1$, so this implies $u'_n u_n^{-1} = 1$ and $u'_n = u_n$.

The unicity implies that the elements $u_n \in U_n(\Lambda)$ must form a compatible system under the maps $U_n(\Lambda) \to U_{n-1}(\Lambda)$, and so determine an element $u \in \varprojlim_n U_n(\Lambda)$. But then $\xi' = \xi^u$ and so

$$[\xi'] = [\xi] \in H^1(G, \varprojlim_n U_n(\Lambda)).$$ This proves that ⊛ is injective.

Claim 2: ⊛ is surjective

Suppose we have a compatible system of elements

$[\xi_n] \in H^1(G, U_n(\Lambda))$.

We claim that these cohomology classes can be represented by a compatible system of cocycles

$\xi_n \in Z^1(G, U_n(\Lambda))$.

We do this recursively, starting with any cocycle $\xi_1 \in Z^1(G, U_1(\Lambda))$ representing $[\xi_1]$.

Let $\xi_2' \in Z^1(G, U_2(\Lambda))$ be any cocycle representing $[\xi_2]$. Since the image $\bar{\bar{\xi}}_2'$ of ξ_2' in $Z^1(G, U_1(\Lambda))$ represents $[\xi_1]$, we know $\xi_1 = \bar{\bar{\xi}}_2'^{\,u_1}$ for some $u_1 \in U_1(\Lambda)$. Lifting u_1 to an element $u_2 \in U_2(\Lambda)$, and setting $\xi_2 := \xi_2'^{\,u_2}$, we see that the image of ξ_2 in $Z^1(G, U_1(\Lambda))$ is ξ_1, and ξ_2 still represents the desired cohomology class.

Iterating this construction gives a compatible system of elements $\xi_n \in Z^1(G, U_n(\Lambda))$ representing the cohomology classes $[\xi_n] \in H^1(G, U_n(\Lambda))$. In the limit, the ξ_n determine a continuous cocycle $\xi \in Z^1(G, \varprojlim_n U_n(\Lambda))$ and it is easy to see that the class of ξ lifts all the initial classes $[\xi_n]$. □

This completes the proof of representability of $H^1(G,U)$. That it is a closed subscheme of $\prod_n H^1(G,V_n)$ follows from a careful inspection of the proof: In the inductive step, we showed that $H^1(G,U_n)$ is noncanonically isomorphic to $\ker(\delta^1) \times H^1(G,V_n)$ in such a way that the projection to $\ker(\delta^1) \subseteq H^1(G,U_{n-1})$ is the map $H^1(G,U_n) \to H^1(G,U_{n-1})$ induced by functoriality. In particular, there is a morphism $\psi_n : H^1(G,U_n) \longrightarrow H^1(G,V_n)$ of functors such that the induced map $H^1(G,U_n) \to H^1(G,U_{n-1}) \times H^1(G,V_n)$ is a closed embedding. Taken together, the maps ψ_n define a morphism

$$H^1(G,U_\bullet) = \varprojlim_n H^1(G,U_n) \longrightarrow \prod_n H^1(G,V_n)$$

which is easily seen to be a closed embedding. $\square$