Lecture 8: Non-abelian cohomology

We've seen that the $\mathbb{Q}_p$-pro-unipotent étale fundamental groupoid $\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}})$ of a smooth variety Y/K comes with a canonical action of the absolute Galois group G_K. In this lecture, we develop a ~~tool~~ for studying actions of groups G on other group(oid)s Π. If Π ~~is an abelian group, then a collection~~ is a topological abelian group, acted on continuously by another topological group G, then a collection of fundamental invariants of the pair (G,Π) is given by the continuous cohomology groups

$$H^i(G,\Pi) \qquad \text{for } i\geq 0.$$

We will explain how to partly generalise these to the case of non-abelian Π.

Suppose that Π and G are topological groups and that G acts continuously on Π from the left.

We define the 0^{th} cohomology group.

$$H^0(G,\Pi) := \Pi^G = \{u \in \Pi : \sigma(u) = u \text{ for all } \sigma \in G\}.$$

There is also the 1^{st} cohomology of Π, defined as follows. A continuous function

$$\xi : G \longrightarrow \Pi$$

is called a (continuous) 1-cocycle just when it satisfies

$$\xi(\sigma\tau) = \xi(\sigma)\cdot\sigma\xi(\tau) \qquad \forall \sigma,\tau \in G.$$

We write $Z^1(G,\Pi)$ for the set of 1-cocycles. There is a natural right action of Π on $Z^1(G,\Pi)$ by

$$u : \xi \longmapsto \xi^u \text{ where } \xi^u(\sigma) := u^{-1}\xi(\sigma)\sigma(u).$$

[Exercise: check that ξ^u is a cocycle.]

The 1^{st} cohomology of Π is defined to be

$$H^1(G,\Pi) := Z^1(G,\Pi)/\Pi.$$

⚠ **Important warning**: Unlike the abelian case, there is not a natural group structure on $H^1(G,\Pi)$. There is, however, a distinguished element $* \in H^1(G,\Pi)$, namely the class of the trivial cocycle $\xi(\sigma)=1 \ \forall \sigma$

Example: Suppose that Π is abelian, and write the group law on Π additively. Then $Z^1(G,\Pi)$ is just the abelian group of 1-cocycles as defined in your favourite book on abelian Galois cohomology. The action of Π on $Z^1(G,\Pi)$ is given by

$$\xi^u(\sigma) = \xi(\sigma) + \sigma(u) - u$$

i.e. $\xi^u = \xi + \partial(\sigma)$ where $\partial : \Pi \longrightarrow Z^1(G,\Pi)$ is the boundary map. So $H^1(G,\Pi) = Z^1(G,\Pi)/\Pi$ is the quotient of $Z^1(G,\Pi)$ by $\mathrm{im}(\partial)$, the subgroup of coboundaries. So $H^1(G,\Pi)$ is the same as <u>defined</u> for abelian Π in this case.

For $i \geq 2$ and Π non-abelian, there is not even a sensible definition of $H^i(G,\Pi)$!

Long exact sequences

Just like for abelian cohomology, a G-equivariant short exact sequence of topological groups induces a long exact sequence in cohomology, though the non-existence of $H^i(G,\Pi)$ for $i \geq 2$ means that these ~~long~~ exact sequences are necessarily not all that long.

Definition: Let

$$1 \longrightarrow \overline{\Pi} \longrightarrow \Pi \longrightarrow \Pi' \longrightarrow 1 \qquad (*)$$

be a short exact sequence of topological groups and continuous group homomorphisms. We say that $(*)$ is *topologically split* just when:

1. the topology on $\overline{\Pi}$ is the subspace topology from Π; and
2. the surjection $\Pi \longrightarrow \Pi'$ admits a continuous splitting (not necessarily compatible with the group law).

G-equivariant topologically split exact sequences induce long exact sequences on cohomology. There are several different versions of this; we state just one.

Theorem: Let

$$1 \longrightarrow A \longrightarrow \Pi \longrightarrow \Pi' \longrightarrow 1$$

be a G-equivariant topologically split central extension of topological groups with a continuous action of G. (So A is abelian.) Then there are coboundary maps

$$\delta^0 : H^0(G,\Pi') \longrightarrow H^1(G,A) \quad \text{[homomorphism of groups]}$$

$$\delta^1 : H^1(G,\Pi') \longrightarrow H^2(G,A) \quad \text{[map of pointed sets]}$$

such that the sequence

$$1 \longrightarrow H^0(G,A) \longrightarrow H^0(G,\Pi) \longrightarrow H^0(G,\Pi') \xrightarrow{\delta^0} H^1(G,A) \longrightarrow H^1(G,\Pi) \longrightarrow H^1(G,\Pi') \xrightarrow{\delta^1} H^2(G,A)$$

is exact. (A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of pointed sets is called <u>exact</u> at Y just when

$$\operatorname{im}(f) = \ker(g) = \{ y \in Y : g(y) = * \}.)$$

Remarks:

1. If Π and Π' are abelian, then δ^0 and δ^1 are the usual coboundary maps in abelian Galois cohomology;
2. The maps δ^0 and δ^1 are natural with respect to morphisms of central extensions in any appropriate sense

Construction of the coboundary maps.

If $u' \in H^0(G, \Pi')$ is a G-invariant element of Π', let $u \in \Pi$ be a preimage of u'. Since u' is G-fixed, we have $u^{-1}\sigma(u) \in A$ for all $\sigma \in G$. The function

$\xi_u : G \longrightarrow A$ given by $\xi_u(\sigma) = u^{-1}\sigma(u)$ is

a continuous 1-cocycle, whose class in $H^1(G, A)$ is independent of the choice of $u \in \Pi$ lifting u'.

The coboundary map

$$\delta^0 : H^0(G, \Pi') \longrightarrow H^1(G, A)$$

is given by $\delta^0(u') := [\xi_u]$.

To define δ^1, suppose that $\xi' : G \longrightarrow \Pi'$ is a continuous 1-cocycle. Since $\Pi \twoheadrightarrow \Pi'$ is topologically split, we can lift ξ' to a continuous function $\xi : G \longrightarrow \Pi$. The fact that ξ' is a 1-cocycle implies that

$$\eta(\sigma,\tau) := \sigma\xi(\tau)\cdot\xi(\sigma\tau)^{-1}\cdot\xi(\sigma) \in A$$

for all $\sigma,\tau \in G$. A rather tedious exercise in symbol-pushing shows that η satisfies the identity

$$\rho\eta(\sigma,\tau)\cdot\eta(\rho\sigma,\tau)^{-1}\cdot\eta(\rho,\sigma\tau)\cdot\eta(\rho,\sigma)^{-1} = 1$$

for all $\rho,\sigma,\tau \in G$, so that $\eta : G\times G \longrightarrow A$ is a continuous 2-cocycle, and that the class of η in $H^2(G,A)$ is independent of the choice of lifting ξ of ξ' and the choice of ξ' in its cohomology class. So we can define

$$\delta^1([\xi']) := [\eta] \in H^2(G,A).$$

This defines all the maps in the long exact sequence. It is left as an exercise to the particularly assiduous reader to verify exactness ~~ex~~ everywhere.

In fact, the long exact sequence associated to a central extension possesses a richer structure than just an exact sequence of groups/pointed sets. If $\xi : G \to \Pi$ and $\alpha : G \to A$ are continuous 1-cocycles, then the pointwise product

$\xi \cdot \alpha : G \to \Pi$ is again a 1-cocycle since

A is central. The map $(\xi, \alpha) \mapsto \xi \cdot \alpha$ induces a map

$$H^1(G, \Pi) \times H^1(G, A) \longrightarrow H^1(G, \Pi)$$

giving a right action of $H^1(G, A)$ on $H^1(G, \Pi)$.

The map $H^1(G, A) \to H^1(G, \Pi)$ induced by functoriality is just the action of $H^1(G, A)$ on the distinguished point $* \in H^1(G, \Pi)$; in particular the long exact sequence gives that the $H^1(G, A)$-orbit of $*$ is exactly $\ker(H^1(G, \Pi) \to H^2(G, A))$.

More generally, we have the following.

Proposition: In the setup of the preceding theorem, the orbits of the $H^1(G,A)$-action on $H^1(G,\Pi)$ are exactly the fibres of $H^1(G,\Pi) \longrightarrow H^1(G,\Pi')$.
So $H^1(G,\Pi)/_{H^1(G,A)} = \operatorname{im}(H^1(G,\Pi) \to H^1(G,\Pi')) = \ker(\delta^1)$

Proof: Let $\xi_1, \xi_2 \in Z^1(G,\Pi)$. If $[\xi_1]$ and $[\xi_2]$ lie in the same $H^1(G,A)$-orbit, then there is a 1-cocycle $\beta \in Z^1(G,A)$ and $u \in \Pi$ such that

$$\xi_2 = \xi_1^u \cdot \beta. \quad (*)$$

So if $\xi_1', \xi_2' \in Z^1(G,\Pi')$ and $u' \in \Pi'$ denote the images in Π', then we have

$$\xi_2' = \xi_1'^{u'} \quad (**)$$

so $[\xi_2'] = [\xi_1']$, i.e. $[\xi_2]$ and $[\xi_1]$ ~~have the same~~ lie in the same fibre of $H^1(G,\Pi) \longrightarrow H^1(G,\Pi')$.
Conversely, if $[\xi_2]$ and $[\xi_1]$ lie in the same fibre, then there is some $u' \in \Pi'$ such that $(**)$ holds. Letting $u \in \Pi$ be a lift of u', there is a unique continuous function $\beta : G \longrightarrow A$ such that $(*)$ holds. One checks by hand that β is a 1-cocycle, so $[\xi_2]$ and $[\xi_1]$ lie in the same orbit.

Serre twisting:

By the preceding proposition, we understand well the orbits of the action of $H^1(G,A)$ on $H^1(G,\Pi)$. We can equally ask about the stabilisers. For the distinguished point $* \in H^1(G,\Pi)$, we have

$$\mathrm{Stabiliser}(*) = \ker\left(H^1(G,A) \longrightarrow H^1(G,\Pi)\right)$$
$$= \mathrm{im}\left(\delta^0 : H^0(G,\Pi') \longrightarrow H^1(G,A)\right)$$

by the long exact sequence.

To access the other stabilisers, we need a construction known as <u>Serre twisting</u>.

Let Π be a topological group with a continuous action of G, and let $\xi \in Z^1(G,\Pi)$ be a 1-cocycle. We can use ξ to define a twisted action of G on Π, given by

$$\sigma * u := \xi(\sigma) \cdot \sigma(u) \cdot \xi(\sigma)^{-1}.$$

(This is again a continuous action by group automorphisms

<u>Definition</u>: The <u>Serre twist</u> ${}_{\xi}\Pi$ is the same topological group Π endowed with the ξ-twisted G-action above.

Proposition: In the above setup, there is a canonical ~~isomorphism~~ bijection

$$H^1(G, {}_\xi\Pi) \xrightarrow{\sim} H^1(G, \Pi).$$

Proof: If $\xi' : G \to \Pi$ is a 1-cocycle for ${}_\xi\Pi$, then the pointwise product $\xi' \cdot \xi$ is a 1-cocycle for Π. The map

$$\begin{aligned} H^1(G, {}_\xi\Pi) &\longrightarrow H^1(G, \Pi) \\ [\xi'] &\longmapsto [\xi'\xi] \end{aligned}$$

is the desired bijection (details left to the reader). $\square$

The Serre twisting bijection does not preserve the distinguished basepoints; in fact, it takes $* \in H^1(G, {}_\xi\Pi)$ to $[\xi] \in H^1(G, \Pi)$. This means that, often, ~~one can prove something for~~ to prove something for a general element of $H^1(G, \Pi)$, it suffices to prove it for the trivial cocycle, and then use Serre twisting to pass to a general ~~co~~cohomology class.

Example: If Π is abelian, then the ~~Serre~~ twisted G-action on Π is the same as the original action. The Serre twisting bijection $H^1(G, \Pi) = H^1(G, {}_\xi\Pi) \cong H^1(G, \Pi)$ is the map given by addition by $[\xi]$.

Now returning to the setup of a G-equivariant topologically split central extension

$$1 \longrightarrow A \longrightarrow \Pi \longrightarrow \Pi' \longrightarrow 1, \qquad (*)$$

suppose that $\xi \in Z^1(G,\Pi)$ is a 1-cocycle. The ξ-twisted action on Π agrees with the original action on A (as A is central), so A is also a G-stable ~~subgro~~ central subgroup of ${}_\xi\Pi$, and the quotient ${}_\xi\Pi/A$ is ${}_\xi\Pi'$, the Serre twist of Π' by the image of ξ in $Z^1(G,\Pi')$. So we have a ξ-twisted ~~exa~~ central extension

$$1 \longrightarrow A \longrightarrow {}_\xi\Pi \longrightarrow {}_\xi\Pi' \longrightarrow 1 \qquad {}_\xi(*)$$

It is easy to check that the Serre twisting bijection $H^1(G,{}_\xi\Pi) \cong H^1(G,\Pi)$ is equivariant for the action of $H^1(G,A)$. This implies

<u>Proposition</u>: For any $\xi \in Z^1(G,\Pi)$, the stabiliser of $[\xi] \in H^1(G,\Pi)$ under the action of $H^1(G,A)$ coming from $(*)$ is equal to the image of the coboundary map $\delta^0 : H^0(G,{}_\xi\Pi') \longrightarrow H^1(G,A)$ from ${}_\xi(*)$.

<u>Proof</u>: True for $\xi = *$ by LES. Serre twist for general ξ, Π

G-equivariant groupoids

Non-abelian cohomology can be used to study actions of groups on groupoids. Let Π be a connected groupoid in topological spaces, and suppose that each $\Pi(x,y)$ is endowed with a continuous action by a ~~grou~~ topological group G, compatibly with path-composition (+ identities, inversion).

If we fix a base vertex $x_0 \in V(\Pi)$, then the action on any $\Pi(x_0, y)$ is determined by the action of G on $\Pi(x_0)$ and on any chosen ~~path~~ element $\gamma_0 \in \Pi(x_0, y)$. Indeed, a general element of $\Pi(x_0,y)$ is $\gamma_0 \cdot u$ for $u \in \Pi(x_0)$ and

$$\sigma(\gamma_0 \cdot u) = \sigma(\gamma_0)\cdot\sigma(u)$$

for all $\sigma \in G$. If γ_0 is G-fixed, then the identification $\Pi(x_0) \cong \Pi(x_0, y)$ given by composition with γ_0 is G-equivariant. In general, the failure of γ_0 to be G-fixed is measured by the function

$$\xi_{\gamma_0} : G \longrightarrow \Pi(x_0) \quad \text{given by } \sigma \longmapsto \gamma_0^{-1}\sigma(\gamma_0).$$

Lemma: The function ξ_{γ_0} is a continuous 1-cocycle, and its class in $H^1(G, \Pi(x_0))$ is independent of the choice of $\gamma_0 \in \Pi(x_0, y)$.

Proof:

$$\xi_{\gamma_0}(\sigma\tau) = \gamma_0^{-1}\sigma\tau(\gamma_0) = \gamma_0^{-1}\sigma(\gamma_0)\cdot\sigma(\gamma_0^{-1}\tau(\gamma_0))$$
$$= \xi_{\gamma_0}(\sigma)\cdot\sigma\xi_{\gamma_0}(\tau).$$

So ξ_{γ_0} is a cocycle. If $\gamma_1 \in \Pi(x_0, y)$ is another path, then $\gamma_1 = \gamma_0 u$ for some $u \in \Pi(x_0)$, so

$$\xi_{\gamma_1}(\sigma) = u^{-1}\gamma_0^{-1}\sigma(\gamma_0)\sigma(u) = \xi_{\gamma_0}^u(\sigma)$$

so $[\xi_{\gamma_1}] = [\xi_{\gamma_0}]$. $\square$

Definition: The function

$$j : V(\Pi) \longrightarrow H^1(G, \Pi(x_0))$$
$$y \longmapsto [\xi_{\gamma_0}]$$

is called the <u>abstract non-abelian Kummer map</u> associated to Π.

Remark: The abstract non-abelian Kummer map measures the failure of $\Pi(x_0, y)$ to have a G-fixed point: $j(y) = *$ iff $\Pi(x_0, y)^G \neq \emptyset$.

Remark: Let Π and G be topological groups, with G acting continuously on Π from the left. A (continuous) left G-action and a right Π-action on a topological space $\underline{P}$ are called compatible just when

$$\sigma(\gamma \cdot u) = \sigma(\gamma)\cdot\sigma(u)$$

for all $\sigma\in G$, $u \in \Pi$, $\gamma \in \underline{P}$. $\underline{P}$ is called a G-equivariant right Π-torsor just when $\underline{P}\neq\emptyset$ and the map

$$\begin{aligned} \underline{P}\times\Pi &\longrightarrow \underline{P}\times\underline{P} \\ (\gamma,u) &\longmapsto (\gamma,\gamma\cdot u) \end{aligned}$$

is a homeomorphism. The construction we gave for groupoids generalizes naturally to show that any G-equivariant right Π-torsor has an associated cohomology class $[\underline{P}]\in H^1(G,\Pi)$, namely the class of the cocycle $\xi(\sigma) = \gamma_0^{-1}\sigma(\gamma_0)$ for some $\gamma_0\in\underline{P}$. In fact this construction gives a bijection

$$\{G\text{-equivariant right } \Pi\text{-torsors}\}/_{\text{iso}} \cong H^1(G,\Pi).$$