# Lecture 7: Galois action on the étale fundamental groupoid

Let $K$ be a field of characteristic $0$ with algebraic closure $\bar{K}$, and let $Y/K$ be a smooth variety. If $x, y \in Y(K)$ are two $K$-rational points, then they determine $\bar{K}$-valued geometric points $x, y \in Y_{\bar{K}}(\bar{K})$, which we continue to denote with the same symbols. (Some authors would write $\bar{x}, \bar{y}$ for these geometric points.)

The absolute Galois group $G_K = \mathrm{Gal}(\bar{K}/K)$ acts on the ensemble $(Y_{\bar{K}}; x, y)$ in a natural way, and this determines (via an appropriate functoriality property) an action of $G_K$ on $\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}}; x, y)$, the $\mathbb{Q}_p$-pro-unipotent étale path-space from $x$ to $y$ in $Y_{\bar{K}}$.

In this lecture, we will survey several important properties of this action, to do with <u>continuity</u>, <u>ramification</u> and <u>purity</u>.

## Continuity:

**Theorem:**

$\mathrm{Lie}(\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}};x))$ and $\mathbb{Q}_p[\![\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}};x,y)]\!]$ are pro-continuous representations of $G_K$ for all $x,y \in Y(\bar{K})$, i.e. they are cofiltered limits of finite-dimensional continuous representations of $G_K$.

---

One consequence of this theorem plays a foundational role in the next section. If $\Lambda$ is a $\mathbb{Q}_p$-algebra, then we topologise $\Lambda$ by giving each finite-dimensional $\mathbb{Q}_p$-subspace its natural p-adic topology, and endowing $\Lambda$ with the inductive limit topology over its finite-dimensional subspaces. This makes $\Lambda$ into a topological ring, and so induces a natural topology on $\cancel{Y}Z(\Lambda)$ for every affine F-scheme ~~Y~~ Z.

**Corollary:** For all $x,y \in Y(K)$, the $G_K$-action on $\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}};x,y)(\Lambda)$ is continuous $\forall \Lambda$.

[redacted]

<u>Proof of Theorem</u> (sketch).

We will show that the action of $G_K$ on the <u>profinite</u> étale fundamental path space
$$\pi_1^{\text{ét}}(Y_{\bar K}; x, y)$$
is continuous, for the profinite topology on the latter. For this, we need to unpick the definition of the Galois action on $\pi_1^{\text{ét}}(Y_{\bar K}; x, y)$ a little. The tautological action of $G_K$ on $\cancel{Y_{\bar K}}$ $\bar K$ (from the left) induces right actions on $\mathrm{Spec}(\bar K)$ and $Y_{\bar K}$. This in turn induces a right action on the category $\underline{\text{FÉt}}(Y_{\bar K})$, where $\sigma \in G_K$ sends a finite étale covering $Y' \to Y_{\bar K}$ to the pullback
$$\begin{tikzcd} \sigma^* Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d] \\ Y_{\bar K} \arrow[r, "\sigma"] & Y_{\bar K} \end{tikzcd}.$$
Note that $\sigma^* Y' \simeq Y'$ as schemes via the topmost arrow, but not as schemes <u>over $Y_{\bar K}$</u>. For any $K$-rational point $x \in Y(K)$, we have
$$(\sigma^* Y')_x = Y'_x$$
← strictly speaking, is a canonical and coherent isomorphism, rather than ~~the~~ an equality.

So if $\gamma : \omega_x^{\text{ét}} \to \omega_y^{\text{ét}}$ is a natural transformation, we can define $\sigma(\gamma)$ to be the nat. trans. with components $\sigma(\gamma)_{Y'} = \gamma_{\sigma^* Y'}$

Now fix a finite étale covering $Y' \to Y_{\bar{K}}$. We may choose a finite extension $L$ of $K$ inside $\bar{K}$ such that $Y'$ and every geometric point of $Y'_x$ and $Y'_y$ is defined over $L$, i.e. there exists a finite étale covering

$$Y'_0 \to Y_L$$ whose base-change to $\bar{K}$ is $Y' \to Y_{\bar{K}}$ and such that the fibres $Y'_{0,x}$, $Y'_{0,y}$ over the $L$-rational points $x, y \in Y(K) \subset Y(L)$ are disjoint unions of copies of $\mathrm{Spec}(L)$. It follows that for $\sigma \in G_L$ there is a canonical isomorphism $f: Y' \xrightarrow{\sim} \sigma^* Y'$ in $\underline{\mathrm{FÉt}}(Y_{\bar{K}})$ such that the induced maps on the fibres at $x$ and $y$ are both the identity. So if $\gamma : \omega_x^{ét} \to \omega_y^{ét}$ is a natural transformation, then $\gamma_{Y'} = \gamma_{\sigma^* Y'} = \sigma(\gamma)_{Y'}$ by naturality. So $G_L$ preserves the fibres of

$$\pi_1^{ét}(Y_{\bar{K}}; x, y) \longrightarrow \mathrm{Hom}(Y'_x, Y'_y).$$

Since the fibres of these maps form a basis of open subsets of $\pi_1^{ét}(Y_{\bar{K}}; x, y)$ and each of them is stable under an open subgroup of $G_K$, it follows that the action is continuous.

To bootstrap this to continuity of the action on the pro-unipotent fundamental group, use the description

$$\mathbb{Q}_p[\![\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}}; x, y)]\!] = \mathbb{Q}_p[\![\pi_1^{ét}(Y_{\bar{K}}; x, y)]\!]$$ from Mal'cev completion.

<u>Unramifiedness</u>

The second property we will need is a criterion for the Galois action on $\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}}; x, y)$ to be unramified when $K$ is a finite extension of $\mathbb{Q}_\ell$. $(\ell \neq p)$

<u>Theorem</u>: Let $\ell \neq p$.

Suppose that $K$ is a finite extension of $\mathbb{Q}_\ell$ and that $Y/K$ is a smooth <u>proper</u> variety with <u>good reduction</u>. Then for any $x, y \in Y(K)$, the $G_K$ action on $\underset{(\ell \neq p)}{\pi_1^{\mathbb{Q}_p}}(Y_{\bar{K}}; x, y)$ is unramified, i.e. the inertia group acts trivially.

More generally, suppose that $Y/K$ is a smooth variety, and that $Y$ is the generic fibre of an $\mathcal{O}_K$-scheme $\mathcal{Y}/\mathcal{O}_K$ which is the complement of a relative normal crossings divisor $\mathcal{D}$ in a smooth <u>proper</u> $\mathcal{O}_K$-scheme $\mathcal{X}$. Then for any $x, y \in \mathcal{Y}(\mathcal{O}_K) \subseteq Y(K)$, the $G_K$-action on $\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}}; x, y)$ is unramified.

__Remarks:__

1. We will be primarily interested in the case that $Y/K$ is a curve. In this case, $\mathcal{D} \subseteq \mathcal{X}$ being a relative normal crossings divisor just means that $\mathcal{D}$ is étale over $\mathrm{Spec}(\mathcal{O}_K)$. E.g. if $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}_\ell}$ then $\mathcal{D} = \{-1, 0, 1, \infty\}$ is relative normal crossings if $\ell > 2$, not if $\ell = 2$ since the points $-1, 1$ collide in the special fibre.

2. This theorem is ~~the~~ starting point for a number of results proving deep connections between the reduction type of a hyperbolic curve $Y$ and the Galois action on its fundamental group. See e.g. ODA, ASADA–MATSUMOTO–ODA, B.–DOGRA.

3. There is also an $\ell = p$ version of the theorem.

__Theorem'__: Suppose $\ell = p$, and that $Y/K$ is smooth. Then for any $x, y \in Y(K)$, $\mathbb{Q}_p[\![\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}}; x, y)]\!]$ is pro-de Rham, i.e. is a cofiltered limit of finite-dimensional de Rham representations. If $Y$ admits a good model $\mathcal{Y}/\mathcal{O}_K$, and if $x, y \in \mathcal{Y}(\mathcal{O}_K)$, then $\mathbb{Q}_p[\![\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}}; x, y)]\!]$ is pro-crystalline.

We discuss the proof in the $\ell \neq p$ case. We will in fact prove that the $G_k$ action on $\pi_1^{(\ell')}(Y_{\bar{k}}; x, y)$ is unramified, where $\pi_1^{(\ell')}(Y_{\bar{k}}; -, -)$ denotes the maximal pro-prime-to-$\ell$ quotient of the profinite étale fundamental groupoid. This uses the theory of specialisation of the étale fundamental group due to Grothendieck.

<u>Theorem</u> (Grothendieck specialisation): Let $R$ be a complete Noetherian DVR with residue field $k$ of characteristic $p \geq 0$. Let $\mathcal{X} \to S = \operatorname{Spec} R$ be a smooth proper $R$-scheme, let $\mathcal{D} \subseteq \mathcal{X}$ be a relative normal crossings divisor, and let $\mathcal{Y} := \mathcal{X} \setminus \mathcal{D}$. Let $\xi$ denote the closed point of $S$ and $\mathcal{Y}_\xi$ the special fibre. Then the inclusion $\mathcal{Y}_\xi \hookrightarrow \mathcal{Y}$ induces an equivalence of categories

$$\mathrm{F\acute{E}t}(\mathcal{Y})^{(\ell')} \xrightarrow{\sim} \mathrm{F\acute{E}t}(\mathcal{Y}_\xi)^{(\ell')}$$

finite étale covers dominated by a prime-to-$\ell$ Galois cover

If moreover $k$ is separably closed and $\eta$ is a geometric generic point of $S$, then the inclusion $\mathcal{Y}_\eta \hookrightarrow \mathcal{Y}$ induces an equivalence $\mathrm{F\acute{E}t}(\mathcal{Y})^{(\ell')} \xrightarrow{\sim} \mathrm{F\acute{E}t}(\mathcal{Y}_\eta)^{(\ell')}$

Remarks:

1. When $\mathcal{D}=\emptyset$, i.e. $\mathcal{Y}=\mathcal{X}$ is smooth and proper over $S$, this is in SGA1, Exposé X. The general case is rather hard to find a good reference for: ~~one approc~~ it follows from a specialisation result for Kummer étale fundamental groups of log-schemes in the thesis of VIDAL, but this requires some unwinding to get to the statement we want.

2. I believe that the statement should be true if the condition of completeness were relaxed to henselianness, which would simplify the following discussion if true.

---

Let us explain how Grothendieck's specialisation theorem implies the unramifiedness of the Galois action. Let $K$ be a finite extension of $\mathbb{Q}_\ell$ with residue field $k$, let $\bar{K}$ be an algebraic closure of $K$ with residue field $\bar{k}$ (an algebraic closure of $k$) and let $\mathbb{C}_K$ be the $\ell$-adic completion of $\bar{K}$. Let $\breve{\mathcal{O}}_K = \mathcal{O}_K \otimes_{W(k)} W(\bar{k})$, which is (Witt vectors.) a complete Noetherian DVR with residue field $\bar{k}$, and which embeds canonically inside $\mathbb{C}_K$. We let $s$ and $\eta$ be the obvious geometric special and generic points of $S=\mathrm{Spec}(\breve{\mathcal{O}}_K)$, valued in $\bar{k}$ and $\mathbb{C}_K$ respectively.

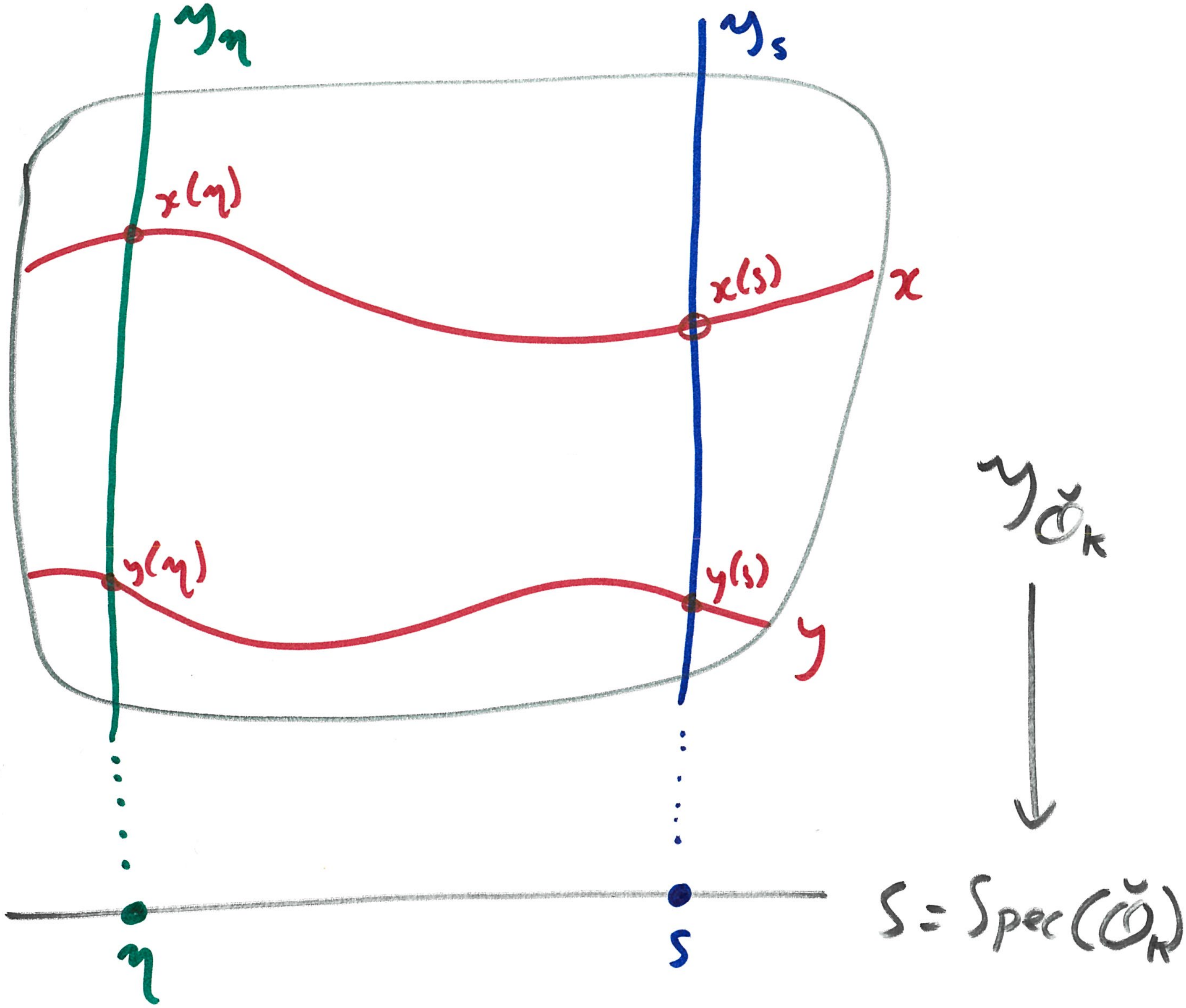
Now Grothendieck's specialization theorem implies that $\pi_1^{(\ell')}(S;s) = \pi_1^{(\ell')}(\mathrm{Spec}(\bar{k});s) = 1$, so the pro-prime-to-$\ell$ fundamental groupoid of $S$ is trivial. [In fact, this is even true without passing to the pro-prime-to-$\ell$ part.]. So there exists a unique path

$$\gamma \in \pi_1^{(\ell')}(S;\eta,s).$$

Now let $\mathcal{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_k)$ be smooth and proper, $\mathcal{D} \subset \mathcal{X}$ a relative normal crossings divisor, $\mathcal{Y} := \mathcal{X} \setminus \mathcal{D}$ and $x, y \in \mathcal{Y}(\mathcal{O}_k)$ two integral points, which we think of as sections of $h: \mathcal{Y} \longrightarrow \mathrm{Spec}(\mathcal{O}_k)$. So $x(\eta)$ and $y(\eta)$ are geometric points of $\mathcal{Y}_{\mathbb{C}_k}$ and $x(s)$, $y(s)$ are geometric points of $\mathcal{Y}_{\bar{k}}$. Combining what we know, we have a sequence of isomorphisms

$$\begin{aligned}
\pi_1^{(\ell')}(\mathcal{Y}_{\bar{K}}; x(\eta), y(\eta)) &\cong \pi_1^{(\ell')}(\mathcal{Y}_{\mathbb{C}_k}; x(\eta), y(\eta)) && \text{[base field invariance of } \pi_1\text{]}\\
&\cong \pi_1^{(\ell')}(\mathcal{Y}_{\breve{\mathcal{O}}_k}; x(\eta), y(\eta)) && \text{[specialization]}\\
&\cong \pi_1^{(\ell')}(\mathcal{Y}_{\breve{\mathcal{O}}_k}; x(s), y(s)) && \text{[compose w. } x(\gamma), y(\gamma)\text{]}\\
&\cong \pi_1^{(\ell')}(\mathcal{Y}_{\bar{k}}; x(s), y(s)) && \text{[specialization]}
\end{aligned}$$

The absolute Galois group acts on everything in sight, making all of these isomorphisms $G_k$-equivariant. But the action of $G_k$ on $\pi_1^{(\ell')}(\mathcal{Y}_{\bar{k}}; x(s), y(s))$ is clearly unramified, as $I_k$ acts trivially on $\mathrm{Spec}(\bar{k})$ and $\mathcal{Y}_{\bar{k}}$. So we are done. $\square$




A cartoon of the specialisation isomorphism

$$\pi_1^{(\ell')}(\mathcal{Y}_\eta; x(\eta), y(\eta)) \cong \pi_1^{(\ell')}(\mathcal{Y}_s; x(s), y(s))$$

## Purity:

The final property of the Galois action we will use is a purity result. Fix distinct prime numbers $\ell \neq p$, and let $K$ be a finite extension of $\mathbb{Q}_\ell$, with $q$ the number of elements of the residue field. We briefly recall the notion of <u>purity</u> of a ($\mathbb{Q}_p$-linear, continuous) representation of $G_K$.

<u>Definition</u>: 1. A <u>$q$-Weil number of weight $n$</u> in a field $E$ of characteristic zero is an element $\alpha \in E$ which is algebraic over $\mathbb{Q}$ and satisfies
$|\iota(\alpha)| = q^{n/2}$ for all complex embeddings
$$\iota : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}.$$
2. An unramified $G_K$-representation $V$ is called <u>pure of weight $n$</u> just when all eigenvalues of a geometric Frobenius $\varphi_K \in G_K$ acting on $V$ are $q$-Weil numbers of weight $n$.

3. More generally, if $V$ is any $G_K$-representation, then Grothendieck's $\ell$-adic monodromy theorem implies that an open subgroup $I_L \leq I_K$ of the inertia group acts unipotently on $V$.

In order to state our purity result, we need to introduce a certain filtration on the fundamental groupoid of $Y_{\bar{k}}$. When $Y_{\bar{k}}$ is proper, this is just the descending central series; in general it is defined as follows.

<u>Definition</u>: Let $Y/k$ be a smooth variety, and write $Y = X \setminus D$ for $X$ smooth and proper and $D$ a reduced normal crossings divisor in $X$. We define the <u>weight filtration</u> $W_\bullet \pi_1^{\mathbb{Q}_p}(Y_{\bar{k}};x)$ as follows:

- $W_{-1}\pi_1^{\mathbb{Q}_p}(Y_{\bar{k}};x)$
- $W_{-2}\pi_1^{\mathbb{Q}_p}(Y_{\bar{k}};x)$ is the kernel of the map
$$\pi_1^{\mathbb{Q}_p}(Y_{\bar{k}};x) \longrightarrow \pi_1^{\mathbb{Q}_p}(X_{\bar{k}};x)^{ab}$$
- For $k \geq 3$, $W_{-k}\pi_1^{\mathbb{Q}_p}(Y_{\bar{k}};x)$ is the subgroup generated by $[W_{-1}, W_{1-k}]$ and $[W_{-2}, W_{2-k}]$.

The weight filtration again has the property that each $W_{-i}$ is normal in $\pi_1^{\mathbb{Q}_p}(Y_{\bar{k}};x)$, all partial quotients $W_{-i}/W_{-i-1}$ are abelian, the ~~core~~ extensions
$$1 \longrightarrow W_{-i}/W_{-i-1} \longrightarrow \pi_1^{\mathbb{Q}_p}/W_{-i-1} \longrightarrow \pi_1^{\mathbb{Q}_p}/W_{-i} \longrightarrow 1$$ are central.

Theorem: The weight filtration on $\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}};x)$ is $G_K$-stable, and $gr_{-i}^W \pi_1^{\mathbb{Q}_p}(Y_{\bar{K}};x) := \dfrac{W_{-i}\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}};x)}{W_{-i-1}\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}};x)}$ is pure of weight $-i$ forall $i \geq 1$.

Remarks:

1. The weight filtration on $\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}};x)$ induces a weight filtration on the group algebra
$$\mathbb{Q}_p[\![\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}};x)]\!]$$
and on
$$\mathbb{Q}_p[\![\pi_1^{\mathbb{Q}_p}(Y_{\bar{K}};x,y)]\!]$$
forall $y$. The graded pieces of this filtration are also pure of weight $-i$.

2. There is also an $\ell=p$ version of this theorem, which we do not state or prove.

---

We describe some of the ideas involved in the proof when $Y=X$ is a smooth projective curve of genus $g$; the general case is B.-LT. We begin with the case $i=1$, where we have
$$\pi_1^{\mathbb{Q}_p}(X_{\bar{K}};x)^{ab} = \mathrm{Ext}^1_{\mathrm{Loc}^{un}_{\mathbb{Q}_p}(X_{\bar{K}},\acute{e}t)}(\underline{\mathbb{Q}_p},\underline{\mathbb{Q}_p})^* \quad \text{[Hurewicz]}$$
$$= H^1_{\acute{e}t}(X_{\bar{K}},\mathbb{Q}_p)^*$$
These identifications are equivariant for the natural $G_K$-action (acting by functoriality), so $gr_{-1}^W\pi_1^{\mathbb{Q}_p}(X_{\bar{K}};x) = \pi_1^{\mathbb{Q}_p}(X_{\bar{K}};x)^{ab}$ is pure of weight $-1$.

How do we describe the next graded piece $gr^W_{-2}\pi_1^{\mathbb{Q}_p}(X_{\bar{K}};x)$?
Well, we know that $\pi_1^{\mathbb{Q}_p}(X_{\bar{K}};x)$ is the $\mathbb{Q}_p$-Malčev completion of the surface group

$$\Pi_g = \langle a_1,\dots,a_g,b_1,\dots,b_g \mid \prod_{i=1}^{g}[a_i,b_i]=1\rangle$$

Here, we have an exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \longrightarrow & \Lambda^2\Pi_g^{ab} & \longrightarrow & gr_\Gamma^2\Pi_g & \to & 0 \\ & & & & a\wedge b & \longmapsto & [a,b] & & \\ & & 1 & \longmapsto & \sum_i a_i\wedge b_i & & & & \end{array}$$

Passing back to the setting of algebraic geometry, this says that the sequence

$$0 \to \mathbb{Q}_p(1) \xrightarrow{\cup^*} \Lambda^2 H^1_{ét}(X_{\bar{K}},\mathbb{Q}_p)^* \longrightarrow gr^W_{-2}\pi_1^{\mathbb{Q}_p}(X_{\bar{K}};x)\to 0$$

is exact, where $\cup^*$ is the dual of the cup product $\cup : \Lambda^2 H^1_{ét}(X_{\bar{K}},\mathbb{Q}_p) \longrightarrow H^2_{ét}(X_{\bar{K}},\mathbb{Q}_p) = \mathbb{Q}_p(-1)$.

Since $\mathbb{Q}_p(1)$ and $\Lambda^2 H^1_{ét}(X_{\bar{K}},\mathbb{Q}_p)^*$ ~~are~~ are pure of weight $-2$, this implies that $gr^W_{-2}\pi_1^{\mathbb{Q}_p}(X_{\bar{K}};x)$ ~~is~~ is pure of weight $-2$.

In the general case, we use the fact, proven by direct calculation, that the $\mathbb{Q}_p$-Mal'čev completion of $\Pi_g$ is the pro-unipotent group associated to the pro-nilpotent Lie algebra

$$\mathfrak{u}_g := \langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g \mid \sum_{i=1}^{g} [\alpha_i, \beta_i] = 0 \rangle$$

⚠ Warning: $\alpha_i \neq \log(a_i)$.

Note that $\mathfrak{u}_g$ is already graded with respect to the grading placing each $\alpha_i, \beta_i$ in degree $-1$.

Back in the setting of algebraic geometry, this implies that $\mathrm{gr}^W_\bullet \mathrm{Lie}(\pi_1^{\mathbb{Q}_p}(X_{\bar{K}};x))$ is the ~~quotient of the~~ free graded pronilpotent Lie algebra generated by $H^1_{ét}(X_{\bar{K}}, \mathbb{Q}_p)^*$ in degree $-1$, quotiented by the ideal generated by the image of $\cup^* : \mathbb{Q}_p(1) \longrightarrow \Lambda^2 H^1_{ét}(X_{\bar{K}}, \mathbb{Q}_p)^*$ in degree $-2$. So we have an exact sequence

$$\mathfrak{f}(1)[2] \longrightarrow \mathfrak{f} \longrightarrow \mathrm{gr}^W_\bullet \mathrm{Lie}(\pi_1^{\mathbb{Q}_p}(X_{\bar{K}};x)) \longrightarrow 0$$

of graded pro-representations of $G_K$, where $\mathfrak{f} = \mathrm{Free}(H^{1*}_{ét})$, and $[2]$ denotes a shift in the grading by $-2$. Since the $i$th graded pieces of $\mathfrak{f}$ and $\mathfrak{f}(1)[2]$ are pure of weight $-i$, this implies the same for $\mathrm{gr}^W_\bullet \mathrm{Lie}(\pi_1^{\mathbb{Q}_p}(X_{\bar{K}};x))$

Remark: In the proof, we are using the fact that the category of pure representations of some weight $i$ is closed under kernels and cokernels in the category of all representations. If we were to restrict to unramified pure representations, then these are even closed under subobjects and quotients. So in this case, the above proof simplifies substantially: once we know that $gr^W_{-1}\pi_1^{\mathbb{Q}_p}(X_{\bar{K}};x) = H^1_{et}(X_{\bar{K}},\mathbb{Q}_p)^*$ is pure of weight $-1$, then we can use the iterated commutator maps

$$\begin{aligned} (gr^W_{-1}\pi_1^{\mathbb{Q}_p}(X_{\bar{K}};x))^{\otimes i} &\twoheadrightarrow gr^W_{-i}\pi_1^{\mathbb{Q}_p}(X_{\bar{K}};x) \\ u_1\otimes\ldots\otimes u_i &\longmapsto [u_1,[\ldots,u_i]\ldots] \end{aligned}$$

to deduce that $gr^W_{-i}\pi_1^{\mathbb{Q}_p}(X_{\bar{K}};x)$ is pure of weight $-i$.