Lecture 6: Étale $\mathbb{Q}_p$-local systems

Now we're going to define the appropriate category of "local systems" on a variety to plug into the Tannakian formalism. As usual, this necessitates working in the étale topology rather than the Zariski topology.

Perhaps our first guess for what should be meant by a "local system" is a locally constant sheaf on the small étale site $X_{ét}$, but this is not the right definition. As an illustration of the problem, one of the most basic $\mathbb{Q}_p$-local systems we would like our formalism to capture is the local system $\mathbb{Q}_p(1)$ on $\mathrm{Spec}(K)$ for a field K of characteristic 0. We'd like to define this as the sheaf

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_n \mu_{p^n} \quad \text{on } \mathrm{Spec}(K)_{ét}$$

but the inverse limit $\varprojlim_n \mu_{p^n}$ is actually 0 in many cases of interest (e.g. K a number field, finite extension of $\mathbb{Q}_\ell$, ... [redacted]).

In fact, if every finite extension of K has only finitely many p-power roots of unity, then $\varprojlim \mu_{p^n} = 0$.

As often, our solution to this problem is to define it away.

<u>Definition</u>: Let X be a smooth variety over a characteristic 0 field k.

1. For $r \geq 0$, a <u>$\mathbb{Z}/p^r\mathbb{Z}$-local system</u> on $X_{\text{ét}}$ is a sheaf of $\mathbb{Z}/p^r\mathbb{Z}$-modules on $X_{\text{ét}}$ which is, locally on $X_{\text{ét}}$, isomorphic to a constant sheaf $\underline{M}$ for some finitely generated $\mathbb{Z}/p^r\mathbb{Z}$-module M.

2. A <u>$\mathbb{Z}_p$-local system</u> on $X_{\text{ét}}$ is a diagram of abelian sheaves

$$\cdots \to E_n \longrightarrow E_{n-1} \to \cdots \to E_2 \to E_1 \to E_0 = 0$$

where each E_r is a $\mathbb{Z}/p^r\mathbb{Z}$-local system and the induced maps $\mathbb{Z}/p^{r-1}\mathbb{Z} \otimes_{\mathbb{Z}/p^r\mathbb{Z}} E_r \longrightarrow E_{r-1}$ are isomorphisms for all $r \geq 1$.

The category of $\mathbb{Z}_p$-local systems on $X_{\text{ét}}$ is an essentially small $\mathbb{Z}_p$-linear abelian $\otimes$-category

3. A <u>$\mathbb{Q}_p$-local system</u> on $X_{ét}$ is a formal symbol "$\mathbb{Q}_p \underset{\mathbb{Z}_p}{\otimes} E$" for E a $\mathbb{Z}_p$-local system. Morphisms of $\mathbb{Q}_p$-local systems are defined by

$$\mathrm{Hom}(\mathbb{Q}_p \otimes E_1, \mathbb{Q}_p \otimes E_2) := \mathbb{Q}_p \underset{\mathbb{Z}_p}{\otimes} \mathrm{Hom}(E_1, E_2).$$

4. A <u>unipotent $\mathbb{Q}_p$-local system</u> on $X_{ét}$ is a $\mathbb{Q}_p$-local system E which is unipotent: it has a filtration

$$0 = \mathrm{Fil}_{-1} E \leq \mathrm{Fil}_0 E \leq \ldots \leq \mathrm{Fil}_n E = E$$

by $\mathbb{Q}_p$-local systems, whose graded pieces are isomorphic to direct sums of copies of the unit local system $\underline{\mathbb{Q}_p}$ "$= \mathbb{Q}_p \underset{\mathbb{Z}_p}{\otimes}$ "$\varprojlim_r \underline{\mathbb{Z}/p^r\mathbb{Z}}$"".

<u>Remarks</u>: 1. The definition of $\mathbb{Q}_p$-local systems above is not the correct one when X is singular, since $\mathbb{Q}_p$-local systems as defined in 3. do not glue in the étale topology when X is singular (they do not form a stack).

2. There is an alternative, more modern approach to defining local systems using the pro-étale topology of BHATT–SCHOLZE.

The categories $\mathrm{Loc}_{\mathbb{Q}_p}(X_{\text{ét}})$ and $\mathrm{Loc}^{\mathrm{un}}_{\mathbb{Q}_p}(X_{\text{ét}})$ of (unipotent) $\mathbb{Q}_p$-local systems on $X_{\text{ét}}$ are pre-Tannakian (they are neutral Tannakian if X is connected). If x is a geometric point of X, then there is an associated fibre functor

$$\omega_x^{\text{ét}} : \mathrm{Loc}^{\mathrm{un}}_{\mathbb{Q}_p}(X_{\text{ét}}) \longrightarrow \underline{\mathrm{Vec}}_{\mathbb{Q}_p}$$

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \text{"}\varprojlim_i E_i\text{"} \longmapsto \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varprojlim_i E_{i,x},$$

where $E_{i,x}$ denotes the fibre/germ of the locally constant sheaf E_i at x.

Definition: The $\mathbb{Q}_p$-pro-unipotent étale fundamental groupoid of a smooth K-variety is, by definition, the Tannakian fundamental groupoid of

$\mathrm{Loc}^{\mathrm{un}}_{\mathbb{Q}_p}(X_{\bar{K},\text{ét}})$, based at the fibre functors coming from geometric points of $X_{\bar{K}}$.

We denote this fundamental groupoid by

$$\pi_1^{\mathbb{Q}_p}(X_{\bar{K}}; -,-).$$

Of course, we should check this agrees with our previous definition.

To make the connection, we need the

Lemma (Descent for finite étale coverings): Let X be a scheme. Then the functor

$$\mathrm{F\acute{E}t}_X \longrightarrow \left(\begin{array}{l}\text{locally constant sheaves}\\ \text{of finite sets on } X_{\acute{e}t}\end{array}\right)$$

$$(X' \to X) \longmapsto \mathrm{Hom}(-, X')$$

is an equivalence of categories.

Proof sketch: First we argue why $\mathrm{Hom}(-,X')$ is a locally constant sheaf. That it is a sheaf follows from the fact that $X' \to X$ is étale and the étale topology is subcanonical. To show local constancy, assume that $X' \to X$ has constant degree d (for simplicity). Let $X'' \to X$ be the degree $d!$ covering parametrising orderings of the fibres of $X' \to X$, i.e. X'' is the complement in

$$\underbrace{X' \times_X X' \times_X \dots \times_X X'}_{d \text{ times}}$$

of all ~~of~~ of the maps

$$\underbrace{X' \times_X \dots \times_X X'}_{d-1 \text{ times}} \longrightarrow \underbrace{X' \times_X \dots \times_X X'}_{d \text{ times}}$$

induced by the diagonal.

By construction, X'' comes with d maps $\phi_1, \dots, \phi_d : X'' \longrightarrow X'$, giving rise to d splittings $s_1, \dots, s_d$ of the pulled-back covering

$$X'' \times_X X' \longrightarrow X''. \quad \text{of degree } d.$$

The images of these splittings are disjoint, so we see $X'' \times_X X' \cong \underbrace{X'' \sqcup X'' \sqcup X'' \sqcup \dots \sqcup X''}_{d \text{ times}}$ with the obvious projection to X''.

It follows that $\mathrm{Hom}(-, X')\big|_{X''} \cong \underline{\{1, \dots, d\}}$ is the constant sheaf on a d-element set. Since $X'' \longrightarrow X$ is surjective, this means that $\mathrm{Hom}(-, X')$ is locally constant.

In the converse direction, suppose that $\mathcal{F}$ is a locally constant sheaf of finite sets, and let $(U_i \longrightarrow X)$ be an étale cover such that each $\mathcal{F}|_{U_i}$ is a constant sheaf of finite sets. So $\mathcal{F}|_{U_i}$ is certainly representable by a scheme finite étale over U_i, namely a finite disjoint union of copies of U_i.

Corollary: Let X be a connected smooth variety over a field of characteristic 0 and let x be a geometric point. Then there are equivalences of categories

$$\begin{pmatrix}\mathbb{Z}/p^i\mathbb{Z}\text{-local systems}\\ \text{on } X_{\acute{e}t}\end{pmatrix} \xrightarrow{\sim} \begin{pmatrix}\text{continuous representations}\\ \text{of } \pi_1^{\acute{e}t}(X;x) \text{ on } \mathbb{Z}/p^i\mathbb{Z}\text{-}\\ \text{modules of finite type}\end{pmatrix}$$

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Proof: We prove the first part; the rest is formal. The category of $\mathbb{Z}/p^i\mathbb{Z}$-local systems on $X_{\acute{e}t}$ is, by definition, the category of $\mathbb{Z}/p^i\mathbb{Z}$-modules in the category of locally constant sheaves of finite sets. By the preceding Lemma + Galois theory of coverings, this is equivalent to the category of $\mathbb{Z}/p^i\mathbb{Z}$-modules in the category of finite continuous $\pi_1^{\acute{e}t}(X;x)$-sets, i.e. representations of $\pi_1^{\acute{e}t}(X;x)$ on finite $\mathbb{Z}/p^i\mathbb{Z}$-modules. $\square$

So we have an $\otimes$-equivalence

$$\mathrm{Loc}^{un}_{\mathbb{Q}_p}(X_{\acute{e}t}) \simeq \underline{\mathrm{Rep}}^{un}_{\mathbb{Q}_p}(\pi_1^{\acute{e}t}(X;x))$$

taking the fibre functor at x to the forgetful functor. To complete the proof of the equivalence of our definitions, it suffices to prove the following Tannakian description of Malčev completion.

Proposition: Let Π be a profinite group, $\mathcal{T} = \underline{\mathrm{Rep}}^{un}_{\mathbb{Q}_p}(\Pi)$ the Tannakian category of unipotent representations on Π, and $\omega_x : \mathcal{T} \longrightarrow \underline{\mathrm{Vec}}_{\mathbb{Q}_p}$ the forgetful functor. Then there is a continuous group hom

$\Pi \longrightarrow \pi_1(\mathcal{T};x)(\mathbb{Q}_p)$ making $\pi_1(\mathcal{T};x)(\mathbb{Q}_p)$ into the Malčev completion of Π.

Proof: Let $\Pi_{\mathbb{Q}_p}$ be the Malčev completion. The universal property ensures that any continuous hom $\Pi \longrightarrow U_{n_m}(\mathbb{Q}_p)$ factors uniquely through $\Pi_{\mathbb{Q}_p}$, so any continuous unipotent representation of Π is a representation of $\Pi_{\mathbb{Q}_p}$ in a unique way. $\therefore$ $\underline{\mathrm{Rep}}^{un}_{\mathbb{Q}_p}(\Pi) \simeq \underline{\mathrm{Rep}}_{\mathbb{Q}_p}(\Pi_{\mathbb{Q}_p})$ so taking Tannaka groups we see $\pi_1(\mathcal{T};x) \simeq \Pi_{\mathbb{Q}_p}$ $\square$.