Lecture 5: The Tannakian formalism

Now we come to our second construction of the $\mathbb{Q}_p$-pro-unipotent étale fundamental groupoid. Here, the idea is as follows. The profinite étale fundamental groupoid was defined purely in terms of the category $\mathrm{FÉt}(X)$ of finite étale coverings + fibre functors. So to "$\mathbb{Q}_p$-linearise" the fundamental groupoid, one approach would be to first "$\mathbb{Q}_p$-linearise the category $\mathrm{FÉt}(X)$ of coverings, and then extract a groupoid out of this category. In the setting of topology, there is a natural candidate for what we should mean by the "$\mathbb{Q}$-linearisation" of the category $\mathrm{Cov}(X)$, namely the category $\underline{\mathrm{Loc}}_{\mathbb{Q}}(X)$ of $\mathbb{Q}$-local systems on X, i.e. sheaves of $\mathbb{Q}$-vector spaces on X which, locally on X, are isomorphic to $\underline{\mathbb{Q}}^{\oplus r}$ for some r (locally constant on X). Or, even more restrictively, one could look only at unipotent local systems, i.e. local systems $\mathbb{E}$ which admit a filtration

$$0 = \mathrm{Fil}_{-1} E \leq \mathrm{Fil}_0 E \leq \dots \leq \mathrm{Fil}_m E = E$$

with each $\mathrm{Fil}_i E / \mathrm{Fil}_{i-1} E$ isomorphic to a direct sum of copies of the constant sheaf $\underline{\mathbb{Q}}$ on X.

So our strategy to define the $\mathbb{Q}_p$-pro-unipotent étale fundamental groupoid consists of two steps:

I. Define, for a smooth variety X in characteristic 0, a category $\underline{\mathrm{Loc}^{\mathrm{un}}_{\mathbb{Q}_p}}(X_{\bar{K},\acute{e}t})$ of <u>unipotent $\mathbb{Q}_p$-local systems on $X_{\bar{K}}$</u> in the étale topology.

II. Extract a fundamental groupoid from this in some formal way.

Today, we focus on the second step, which is afforded by a formalism known as the ~~Tannakian fo~~ <u>TANNAKIAN formalism</u>

Pre-Tannakian categories

Fix F a field of characteristic 0 (e.g. $F = \mathbb{Q}_p$).

An $\underline{F\text{-linear abelian } \otimes\text{-category}}$ is a $\otimes$-category $\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{1})$ in which each Hom-set $\mathrm{Hom}_{\mathcal{T}}(E, E')$ has been given the structure of an F-vector space so that the composition and tensoring maps

$$\mathrm{Hom}_{\mathcal{T}}(E', E'') \times \mathrm{Hom}_{\mathcal{T}}(E, E') \longrightarrow \mathrm{Hom}_{\mathcal{T}}(E, E'')$$

$$\mathrm{Hom}_{\mathcal{T}}(E_1, E_1') \times \mathrm{Hom}_{\mathcal{T}}(E_2, E_2') \longrightarrow \mathrm{Hom}_{\mathcal{T}}(E_1 \otimes E_2, E_1' \otimes E_2')$$

are F-bilinear.

If E is an object in such a category, by a $\underline{\text{strong dual}}$ (or just $\underline{\text{dual}}$) of E we mean an object $E^* \in \mathcal{T}$ endowed with maps

$\mathrm{ev} : E^* \otimes E \longrightarrow \mathbb{1}$ "evaluation"

$\delta : \mathbb{1} \longrightarrow E \otimes E^*$ "coevaluation"

making the triangles

$$\begin{tikzcd} E \arrow[r, "\delta \otimes 1"] \arrow[dr, equal] & E \otimes E^* \otimes E \arrow[d, "1 \otimes \mathrm{ev}"] \\ & E \end{tikzcd}$$

and

$$\begin{tikzcd} E^* \arrow[r, "1 \otimes \delta"] \arrow[dr, equal] & E^* \otimes E \otimes E^* \arrow[d, "\mathrm{ev} \otimes 1"] \\ & E^* \end{tikzcd}$$

commute.

<u>Example</u>: If $\mathcal{T} = \underline{\mathrm{Mod}}_F$ is the category of F-vector spaces (with $\otimes = \otimes_F$, $\mathbb{1} = F$), then $V \in \underline{\mathrm{Mod}}_F$ has a strong dual if and only if V is finite-dimensional. The strong dual to V is the usual dual $V^* = \mathrm{Hom}_F(V, \mathbb{1})$, the evaluation map $ev: V^* \otimes V \longrightarrow \mathbb{1}$ is given by $\psi \otimes v \longmapsto \psi(v)$, and the coevaluation $\delta: \underline{\mathbb{1}} \longrightarrow V \otimes V^*$ sends $1 \in F = \mathbb{1}$ to $\sum_i v_i \otimes \psi_i$, where $(v_i)_{i \in I}$ is a basis for V and $(\psi_i)_{i \in I}$ is the dual basis.

A strong dual (E^*, ev, δ) of an object $E \in \mathcal{T}$, if it exists, is unique up to unique isomorphism.
We say that $\mathcal{T}$ is <u>rigid</u> just when every object has a dual.

<u>Definition</u>: A <u>pre-Tannakian category</u> (over F) is ~~a rigid F-l~~ a small rigid F-linear abelian $\otimes$-category.

<u>Example</u>: $\mathbb{Q}$-local systems on a topological space form a pre-Tannakian category over $\mathbb{Q}$. Duals are the usual sheaf-theoretic duals: $E^* = \underline{\mathrm{Hom}}_{\mathbb{Q}}(E, \underline{\mathbb{Q}})$ where $\underline{\mathbb{Q}} = \mathbb{1}$ is the constant sheaf.

<u>Fibre functors</u>: Let $\underline{\mathrm{Vec}}_F$ denote the $\otimes$-category of finite-dimensional vector spaces. A (neutral) <u>fibre functor</u> on a pre-Tannakian category $\mathcal{T}$ is an exact F-linear $\otimes$-functor

$$\omega_x : \mathcal{T} \longrightarrow \underline{\mathrm{Vec}}_F .$$

⚠ We do not – yet – require that fibre functors be faithful.

We usually write fibre functors as $\omega_x, \omega_y, \omega_z, \ldots$ and adopt the shorthand

$$E_x := \omega_x(E). \leftarrow$$ supposed to look like taking a fibre of a sheaf at a point.

Tannakian fundamental groupoids

Let $\mathcal{T}$ be pre-Tannakian and let ω_x, ω_y be fibre functors. A natural transformation $\gamma : \omega_x \longrightarrow \omega_y$ is called **$\otimes$-natural** just when the square

$$\begin{tikzcd}
E_x \otimes E'_x \arrow[r, "\gamma_E \otimes \gamma_{E'}"] \arrow[d, "\wr", no head, equal] & E_y \otimes E'_y \arrow[d, "\wr", no head, equal] \\
(E \otimes E')_x \arrow[r, "\gamma_{E \otimes E'}"] & (E \otimes E')_y
\end{tikzcd}$$

commutes for all $E, E' \in \mathcal{T}$, as does the square

$$\begin{tikzcd}
\mathbb{1}_x \arrow[r, "\gamma_{\mathbb{1}}"] \arrow[d, "\wr", no head, equal] & \mathbb{1}_y \arrow[d, "\wr", no head, equal] \\
F \arrow[r, equal] & F
\end{tikzcd}$$

More generally, if Λ is an F-algebra, then we write $\omega_{x,\Lambda}$ for the $\otimes$-functor given by the composite

$$\begin{array}{c}
\mathcal{T} \xrightarrow{\omega_x} \underline{\mathrm{Vec}}_F \longrightarrow \underline{\mathrm{Mod}}_\Lambda \\
\qquad\quad V \longmapsto \Lambda \otimes_F V
\end{array}$$

We say that a natural transformation $\omega_{x,\Lambda} \xrightarrow{\gamma} \omega_{y,\Lambda}$ is **$\otimes$-natural** when it satisfies the obvious Λ-linear analogues of the above conditions.

Proposition/Theorem

1. Every $\otimes$-natural transformation $\omega_{x,\Lambda} \longrightarrow \omega_{y,\Lambda}$ is a $\otimes$-natural isomorphism. $(\forall\, \mathcal{T}, \omega_x, \omega_y, \Lambda)$

2. The functor

$$\underline{\mathrm{Alg}}_F \longrightarrow \underline{\mathrm{Set}} \quad \text{given by} \quad \Lambda \longmapsto \underbrace{\mathrm{Iso}^{\otimes}(\omega_{x,\Lambda}, \omega_{y,\Lambda})}_{\text{set of } \otimes\text{-natural isos.}} = \mathrm{Hom}^{\otimes}(\omega_{x,\Lambda}, \omega_{y,\Lambda})$$

is representable by an affine F-scheme

$$\pi_1(\mathcal{T}; x, y)$$

called the <u>Tannakian path-space</u>. $(\forall\, \mathcal{T}, \omega_x, \omega_y)$.

3. For varying fibre functors, the maps

$$\pi_1(\mathcal{T}; y, z) \times \pi_1(\mathcal{T}; x, y) \longrightarrow \pi_1(\mathcal{T}; x, z)$$

given by composition of $\otimes$-natural isomorphisms/transformations make the schemes $\pi_1(\mathcal{T}; x, y)$ into a groupoid in affine F-schemes: the <u>Tannakian fundamental groupoid</u> $\pi_1(\mathcal{T})$ of $\mathcal{T}$.

Examples

1. $\mathcal{T} = \underline{\mathrm{Rep}}_F(U)$ for an affine group-scheme U/F,
$\omega_x = \omega_y : \underline{\mathrm{Rep}}_F(U) \longrightarrow \underline{\mathrm{Vec}}_F$ forgetful functor
$\Rightarrow \pi_1(\mathcal{T}; x) \cong U$.

2. $\mathcal{T} = \underline{\mathrm{Loc}}_{\mathbb{Q}}^{un}(X)$ the category of unipotent $\mathbb{Q}$-local systems on a nice connected topological space X,
$\omega_x = \omega_y : \cancel{\underline{\mathrm{Rep}}}\ \underline{\mathrm{Loc}}_{\mathbb{Q}}^{un}(X) \longrightarrow \underline{\mathrm{Vec}}_{\mathbb{Q}}$ fibre functor at a point $x \in X$. Then
$\underline{\mathrm{Loc}}_{\mathbb{Q}}^{un}(X) \cong \underline{\mathrm{Rep}}_{\mathbb{Q}}^{un}(\pi_1(X; x))$ ← category of unipotent representations.
$\Rightarrow \pi_1(\mathcal{T}; x) \cong \pi_1(X; x)_{\mathbb{Q}}$ ← $\mathbb{Q}$-Mal'čev completion

Matrix Coefficients

Tannakian fundamental groupoids can be understood rather explicitly via matrix coefficients.

<u>Definition</u> Let $\mathcal{T}$ be pre-Tannakian, ω_x, ω_y fibre functors. An <u>abstract matrix coefficient</u> for $(\mathcal{T}, \omega_x, \omega_y)$ is a triple (E, v, ϕ) where $E \in \mathcal{T}$, $v \in E_x$ and $\phi \in E_y^*$.

(N.B. $(E_y)^* = (E^*)_y$ since any $\otimes$-functor preserves strong duals.)

Any abstract matrix coefficient (E, v, ϕ) determines a functional $f_{(E,v,\phi)} : \pi_1(\mathcal{T}; x, y) \longrightarrow \mathbb{A}^1_F$ by the composite

$$\pi_1(\mathcal{T}; x, y) \xrightarrow[\text{on } E]{\text{action}} \mathbb{A}(\mathrm{Hom}_F(E_x, E_y)) \xrightarrow[\phi \circ -]{} \mathbb{A}(\mathrm{Hom}_F(E_x, F)) \xrightarrow[\text{at } v]{\text{evaluate}} \mathbb{A}^1_F .$$

<u>Example</u>: If $U \leq GL_r$ is a subgroup, let $\mathcal{T} = \underline{\mathrm{Rep}}_F(U)$ and $\omega_x = \omega_y =$ forgetful functor. Let V be the standard r-dimensional representation of U, $v_j \in V$ the j^{th} basis vector, ϕ_i the i^{th} coordinate projection. Then (V, v_j, ϕ_i) is an abstract matrix coefficient, and $f_{(V, v_j, \phi_i)} : U \to \mathbb{A}^1_F$ is the functional picking out the ijth matrix entry.

Two abstract matrix coefficients (E, v, ϕ) and (E', v', ϕ') are said to be <u>basic equivalent</u> if there exists a morphism $f: E \to E'$ in $\mathcal{T}$ such that $v' = f_*(v)$ and $\phi = f^*(\phi')$. We let $\boxed{H_{x,y}}$ denote the set of abstract matrix coefficients up to the equivalence relation generated by basic equivalence.

We define an addition and multiplication on $H_{x,y}$ by

$$(E_1, v_1, \phi_1) + (E_2, v_2, \phi_2) = (E_1 \oplus E_2, v_1 \oplus v_2, \phi_1 \oplus \phi_2)$$

$$(E_1, v_1, \phi_1) \cdot (E_2, v_2, \phi_2) = (E_1 \otimes E_2, v_1 \otimes v_2, \phi_1 \otimes \phi_2).$$

These are well-defined, and make $H_{x,y}$ into a commutative ring with identities $(\mathbb{1}, 1, 1)$ and $(0, 0, 0)$. ~~and~~
~~Inverses given by~~ $\cancel{(E, v, \phi) \longmapsto (E, -v, \phi)}$

There is a homomorphism $F \to H_{x,y}$ of rings, given by $\lambda \longmapsto (\mathbb{1}, \lambda, 1)$, making $H_{x,y}$ into an F-algebra.

There are further structures on the rings $H_{x,y}$.

1. For fibre functors $\omega_x, \omega_y, \omega_z$, there is a <u>cocomposition map</u>

$$\Delta = \Delta_{x,y,z} : H_{x,z} \longrightarrow H_{y,z} \otimes H_{x,y}$$

given by $\quad (E, v, \phi) \longmapsto \sum_i (E, v_i, \phi) \otimes (E, v, \phi_i)$

where (v_i) is a basis of E_y with dual basis (ϕ_i).

(The map $\Delta_{x,y,z}$ is well-defined, and a homomorphism of F-algebras.)

2. For a fibre functor ω_x, there is a <u>counit map</u>

$$\varepsilon = \varepsilon_x : H_{x,x} \longrightarrow F$$
$$(E, v, \phi) \longmapsto \phi(v)$$

This is a homomorphism of F-algebras.

3. For fibre functors ω_x, ω_y, there is an <u>antipode map</u>

$$S = S_{x,y} : H_{x,y} \longrightarrow H_{y,x}$$
$$(E, v, \phi) \longmapsto (E^*, \phi^*, v^*)$$

This is also a homomorphism of F-algebras.

__Theorem__: Let $\mathcal{T}$ be a pre-Tannakian category. Then for any fibre functors ω_x, ω_y, there is a canonical isomorphism

$$\pi_1(\mathcal{T}; x, y) \cong \mathrm{Spec}(H_{x,y})$$

of affine F-schemes. These isomorphisms are compatible with composition, identities and reversal, so constitute an isomorphism of groupoids

$$\pi_1(\mathcal{T}; -,-) \cong \mathrm{Spec}(H_{-,-}).$$

We will prove this in several steps. The key observation is the following.

__Proposition__: There is a canonical isomorphism

$$H_{x,y}^* \cong \mathrm{Hom}(\omega_x, \omega_y)$$

in pro-$\underline{\mathrm{Vec}}_F$. Here, the profinite-dimensional vector space structure on $\mathrm{Hom}(\omega_x, \omega_y)$ is the natural one coming from writing it as an end

$$\mathrm{Hom}(\omega_x, \omega_y) = \int_{E \in \mathcal{T}} \mathrm{Hom}(E_x, E_y)$$

and taking this end in pro-$\underline{\mathrm{Vec}}_F$.

Proof: For any $E \in \mathcal{T}$, the map

$$E_x \times E_y^* \longrightarrow H_{x,y}$$
$$(v, \phi) \longmapsto (E, v, \phi)$$

is F-bilinear. So given an F-linear map $f : H_{x,y} \longrightarrow F$ we can define an F-linear map $\gamma_E : E_x \longrightarrow E_y$ by specifying that

$$\boxed{\phi(\gamma_E(v)) = f(E, v, \phi) \quad \circledast}$$

for all $v \in E_x$, $\phi \in E_y^*$.

The maps γ_E are components of a natural transformation $\gamma : \omega_x \longrightarrow \omega_y$. For, if $\alpha : E_1 \longrightarrow E_2$ is a morphism in $\mathcal{T}$, then

$$\phi(\alpha_y(\gamma_{E_1}(v))) = f(E_1, v, \alpha_y^* \phi) = f(E_2, \alpha_x(v), \phi)$$
$$= \phi(\gamma_{E_2}(\alpha_x(v)))$$

for all $v \in E_{1,x}$ and all $\phi \in E_{2,y}^*$, so

$$\begin{tikzcd}
E_{1,x} \arrow[r, "\gamma_{E_1}"] \arrow[d, "\alpha_x"] & E_{1,y} \arrow[d, "\alpha_y"] \\
E_{2,x} \arrow[r, "\gamma_{E_2}"] & E_{2,y}
\end{tikzcd}$$

commutes.

Conversely, given a natural transformation $\gamma : \omega_x \longrightarrow \omega_y$, one can define a map $f : H_{x,y} \longrightarrow F$ by $\circledast$, which is well-defined (i.e. invariant under basic equivalences). Since $\gamma_E : E_x \longrightarrow E_y$ is F-linear for all $E \in \mathcal{T}$, it follows that f is F-linear when restricted to the image of $E_x \otimes E_y^* \longrightarrow H_{x,y}$. As these images exhaust $H_{x,y}$, it follows that f is F-linear.

We have thus constructed ~~functions~~ a bijection

$$H_{x,y}^* \cong \mathrm{Hom}(\omega_x, \omega_y),$$

which is clearly F-linear. The assertion that this is an isomorph of pro-finite-dimensional vector spaces amounts to the assertion that if $f \in H_{x,y}^*$ and $\gamma \in \mathrm{Hom}(\omega_x, \omega_y)$ correspond to one another, then $\cancel{\gamma_E : E_x \longrightarrow E_y}$ $\gamma_E \in \mathrm{Hom}(E_x, E_y)$ is determined by the restriction of f to a finite-dimensional subspace of $H_{x,y}^*$. But this is clear: γ_E is determined by the restriction of f to the image of $E_x \otimes E_y^* \longrightarrow H_{x,y}$. $\square$

Next, we describe what the extra structures on $H_{x,y}$ correspond to on the spaces $\mathrm{Hom}(\omega_x, \omega_y)$ of natural transformations. The following is clear/easy.

<u>Lemma:</u>

1. For all fibre functors $\omega_x, \omega_y, \omega_z$, the ~~comultiplication~~ cocomposition

$$\Delta_{x,y,z}: H_{x,z} \longrightarrow H_{y,z} \otimes H_{x,y}$$

is dual to the composition map

$$\mathrm{Hom}(\omega_y, \omega_z) \hat{\otimes} \mathrm{Hom}(\omega_x, \omega_y) \longrightarrow \mathrm{Hom}(\omega_x, \omega_z).$$

2. For all fibre functors ω_x, the counit

$$\varepsilon_x : H_{x,x} \longrightarrow F$$

is dual to the unit map

$$F \longrightarrow \mathrm{Hom}(\omega_x, \omega_x)$$
$$1 \longmapsto 1_{\omega_x}$$

3. For all fibre functors ω_x, ω_y, the antipode

$$S : H_{y,x} \longrightarrow H_{x,y}$$

is dual to the map

$$\mathrm{Hom}(\omega_x, \omega_y) \longrightarrow \mathrm{Hom}(\omega_y, \omega_x)$$
$$(\gamma_E)_E \longmapsto (\gamma_{E^*}^*)_E .$$

For the remaining structures, we need a good description of the completed tensor product $\mathrm{Hom}(\omega_x,\omega_y)\hat{\otimes}\mathrm{Hom}(\omega_x,\omega_y)$.

Let $\omega_x\boxtimes\omega_x : \mathcal{T}\times\mathcal{T}\longrightarrow \underline{\mathrm{Vec}}_F$ be the functor

$$(E_1,E_2)\longmapsto E_{1,x}\otimes E_{2,x}$$

and define $\omega_y\boxtimes\omega_y$ similarly. One checks that there is an isomorphism

$$\mathrm{Hom}(\omega_x,\omega_y)\hat{\otimes}\mathrm{Hom}(\omega_x,\omega_y)\xrightarrow{\sim}\mathrm{Hom}(\omega_x\boxtimes\omega_x,\omega_y\boxtimes\omega_y)$$

in pro-$\underline{\mathrm{Vec}}_F$, sending $\gamma_1\otimes\gamma_2$ to the natural transformation $\gamma:\omega_x\boxtimes\omega_x\longrightarrow\omega_y\boxtimes\omega_y$ with components

$$\gamma_{(E_1,E_2)}=\gamma_{E_1}\otimes\gamma_{E_2}.$$

__Lemma__ (ct'd)

4. For all fibre functors ω_x,ω_y, the multiplication
$$\mu: H_{x,y}\otimes H_{x,y}\longrightarrow H_{x,y}$$
is dual to the map
$$\mathrm{Hom}(\omega_x,\omega_y)\longrightarrow\mathrm{Hom}(\omega_x\boxtimes\omega_x,\omega_y\boxtimes\omega_y)$$
sending $\gamma:\omega_x\longrightarrow\omega_y$ to the natural transformation $\Delta(\gamma)$ with components $\Delta(\gamma)_{(E_1,E_2)}=\gamma_{E_1\otimes E_2}$
5. For all fibre functors ω_x,ω_y, the unit map
$$\eta: F\longrightarrow H_{x,y}$$
is dual to the map $\mathrm{Hom}(\omega_x,\omega_y)\longrightarrow F$ given by the action on $\mathbb{1}$. (Note $F=\mathrm{Hom}(\mathbb{1}F,F)=\mathrm{Hom}(\mathbb{1}_x,\mathbb{1}_y)$).

Now if Λ is an F-algebra, there is a canonical identification

$$\mathrm{Hom}(\omega_x,\omega_y)_\Lambda = \mathrm{Hom}(\omega_{x,\Lambda},\omega_{y,\Lambda})$$

given by the composite

$$\begin{aligned}\mathrm{Hom}(\omega_x,\omega_y)_\Lambda &= \Big(\int_{E\in\mathcal{T}} \mathrm{Hom}_F(E_x,E_y)\Big)_\Lambda \\ &= \int_{E\in\mathcal{T}} \big(\Lambda \otimes_F \mathrm{Hom}_F(E_x,E_y)\big) \\ &= \int_{E\in\mathcal{T}} \mathrm{Hom}_\Lambda(\Lambda\otimes_F E_x, \Lambda\otimes_F E_y) \\ &= \mathrm{Hom}(\omega_{x,\Lambda},\omega_{y,\Lambda}).\end{aligned}$$

<u>Lemma</u>: Under the above ~~correspon~~ identification, the grouplike elements of $\mathrm{Hom}(\omega_x,\omega_y)_\Lambda$ correspond to the $\otimes$-natural transformations $\omega_{x,\Lambda}\longrightarrow\omega_{y,\Lambda}$.

<u>Proof</u>: If $\gamma\in \mathrm{Hom}(\omega_x,\omega_y)_\Lambda \overset{=\mathrm{Hom}(\omega_{x,\Lambda},\omega_{y,\Lambda})}{}$, the image of γ under the comultiplication

$$\Delta : \mathrm{Hom}(\omega_x,\omega_y)_\Lambda \longrightarrow \underbrace{\big[\mathrm{Hom}(\omega_x,\omega_y)_\Lambda \hat{\otimes} \mathrm{Hom}(\omega_x,\omega_y)\big]_\Lambda}_{\shortparallel \atop \mathrm{Hom}(\omega_{x,\Lambda}\boxtimes\omega_{x,\Lambda},\ \omega_{y,\Lambda}\boxtimes\omega_{y,\Lambda})}$$

is the natural transformation with components

$$\Delta(\gamma)_{(E_1,E_2)} = \gamma_{E_1\otimes E_2}.$$

So $\Delta(\gamma) = \gamma \otimes \gamma$ if and only if

$\gamma_{E_1 \otimes E_2} = \gamma_{E_1} \otimes \gamma_{E_2}$ for all E_1, E_2.

Similarly $\varepsilon(\gamma) = 1$ if and only if

$\gamma_{\underline{1}} = 1$.

So γ is grouplike if and only if γ is $\otimes$-natural. $\square$

Proof of Theorem: For any F-algebra Λ, we have

$$\mathrm{Hom}_{\underline{\mathrm{Alg}}_F}(H_{x,y}, \Lambda) \cong \mathrm{Hom}(\omega_x, \omega_y)_\Lambda^{\mathrm{gplike}}$$
$$= \mathrm{Hom}^{\otimes}(\omega_{x,\Lambda}, \omega_{y,\Lambda})$$
$$= \mathrm{Iso}^{\otimes}(\omega_{x,\Lambda}, \omega_{y,\Lambda}).$$

So $\mathrm{Spec}(H_{x,y})$ represents the functor of $\otimes$-natural isomorphisms. $\square$

Remark: The above proof shows the representability of the functor of $\otimes$-natural isomorphisms.

Universal objects

There is an analogue of universal coverings in the Tannakian formalism.

Proposition: Let $\mathcal{T}$ be a pre-Tannakian category and $\omega_x : \mathcal{T} \longrightarrow \underline{\mathrm{Vec}}_F$ a fibre functor. Then ω_x is pro-representable. same pro-representability criterion as usual.

We call the pro-representing pro-object $({}_xE^{\mathcal{T}}, e_x^{\mathcal{T}})$ the <u>universal object</u>.

Example: If $\mathcal{T} = \mathrm{Rep}(U)$, ~~then~~ and ω_x is the forgetful functor, then ${}_xE^{\mathcal{T}} = F[\![U]\!]$, viewed as a pro-finite dimensional $F[\![U]\!]$-module with the left-multiplication action. $e_x^{\mathcal{T}} \in {}_xE^{\mathcal{T}}$ is the identity $1 \in F[\![U]\!]$.

Although we will not use it for a while, we remark that ${}_xE^{\mathcal{T}}$ carries the structure of a cocommutative coalgebra in pro-$\mathcal{T}$ with ~~~~ comultiplication $\Delta : {}_xE^{\mathcal{T}} \longrightarrow {}_xE^{\mathcal{T}} \hat{\otimes} {}_xE^{\mathcal{T}}$ and counit $\varepsilon : {}_xE^{\mathcal{T}} \longrightarrow \underline{1}$ the unique maps such that $\Delta_x(e_x^{\mathcal{T}}) = e_x^{\mathcal{T}} \otimes e_x^{\mathcal{T}}$ and $\varepsilon_x(e_x^{\mathcal{T}}) = 1$.

Neutral Tannakian Categories

Often, it is useful to restrict to pre-Tannakian categories satisfying a certain connectedness condition. If $\mathcal{T}$ is pre-Tannakian and $\omega_x : \mathcal{T} \longrightarrow \underline{\mathrm{Vec}}_F$ is a fibre functor, then the following are equivalent:

1. ω_x is faithful (i.e. $\mathrm{Hom}_{\mathcal{T}}(E_1, E_2) \longrightarrow \mathrm{Hom}_F(E_{1,x}, E_{2,x})$ is injective)
2. ω_x is conservative (i.e. $\omega_x(f)$ is an iso $\Leftrightarrow$ f is an iso)
3. ω_x reflects zero objects (i.e. $E_x = 0 \Leftrightarrow E = 0$).

<u>Definition</u>: $\mathcal{T}$ is called <u>neutral Tannakian</u> if it possesses at least one faithful fibre functor.

<u>Proposition</u>: If $\mathcal{T}$ is neutral Tannakian, then

1. $\pi_1(\mathcal{T}; x, y) \neq \emptyset$ for all fibre functors x, y
2. Every fibre functor is faithful.

<u>Proof (sketch)</u>:

1. It suffices to prove this when ω_x is faithful. Let ${}_xE^{\mathcal{T}}$ denote the universal object and $\varepsilon : {}_xE^{\mathcal{T}} \longrightarrow \mathbb{1}$ the map sending $e_x^{\mathcal{T}}$ to 1.

$${}_xE^{\mathcal{T}} = \text{“}\varinjlim_i\text{”}\, {}_xE_i^{\mathcal{T}}$$

ε, as a morphism of pro-objects, is represented by a morphism ${}_xE^{\mathcal{T}}_{i_0} \longrightarrow \mathbb{1}$ in $\mathcal{T}$ for some i_0.

Since the image of ${}_xE^{\mathcal{T}}_{i_0,x} \longrightarrow \mathbb{1}_x = F$ contains 1 by assumption, it is surjective. ~~More~~ So by faithfulness ${}_xE^{\mathcal{T}}_{i_0} \longrightarrow \mathbb{1}$ is surjective (epimorphic) in $\mathcal{T}$.

More generally, the composite

$$ {}_xE^{\mathcal{T}}_{i} \longrightarrow {}_xE^{\mathcal{T}}_{i_0} \longrightarrow \mathbb{1} $$

is surjective for all $i \longrightarrow i_0$ in I.

Applying ω_y, we see that

$$ {}_xE^{\mathcal{T}}_{i,y} \longrightarrow \mathbb{1}_y = F $$

is surjective for all such i. So

$$ \varepsilon_y : {}_xE^{\mathcal{T}}_y := \varprojlim_{i \in I} {}_xE^{\mathcal{T}}_{i,y} \longrightarrow F \text{ is surjective} $$

(using exactness of cofiltered limits in pro-$\underline{\mathrm{Vec}}_F$) and in particular ${}_xE^{\mathcal{T}}_y \neq 0$. So

$$ \mathrm{Hom}(\omega_x, \omega_y) \underset{\text{Yoneda}}{=} {}_xE^{\mathcal{T}}_y \neq 0 $$

and so $\pi_1(\mathcal{T}; x, y) = \mathrm{Spec}(\mathrm{Hom}(\omega_x, \omega_y)^*) \neq \emptyset$.

2. Follows from 1. that any two fibre functors ω_x, ω_y become isomorphic over a field extension F'/F. So ω_x is faithful $\Longleftrightarrow$ ω_y is faithful.

The big theorem regarding neutral Tannakian categories is the following.

<u>Theorem</u> (Tannakian reconstruction) Let $\mathcal{T}$ be a neutral Tannakian category and ω_x a fibre functor. Then $\mathcal{T}$ is canonically $\otimes$-equivalent to $\underline{\mathrm{Rep}}_F(\pi_1(\mathcal{T};x))$, in such a way that the fibre functor ω_x corresponds to the forgetful functor $\underline{\mathrm{Rep}}_F(\pi_1(\mathcal{T};x)) \longrightarrow \underline{\mathrm{Vec}}_F$.

<u>Proof</u> (omitted).

Theorem (HUREWICZ theorem for unipotent Tannakian categories). Let $\mathcal{T}$ be a unipotent neutral Tannakian category. Then $\pi_1(\mathcal{T};x)^{ab}$ is canonically isomorphic to the pro-vector group associated to $\mathrm{Ext}^1_{\mathcal{T}}(\mathbb{1},\mathbb{1})^*$.

Proof (sketch) Since $\pi_1(\mathcal{T};x)^{ab}$ is a pro-vector group, it suffices to show that there is a canonical F-linear isomorphism

$$\mathrm{Ext}^1_{\mathcal{T}}(\mathbb{1},\mathbb{1}) \cong \mathrm{Hom}(\pi_1(\mathcal{T};x),\mathbb{G}_a)$$

In one direction, if

$$0 \to \mathbb{1} \longrightarrow E \longrightarrow \mathbb{1} \longrightarrow 0$$

is an extension, then we may pick a basis of E_x for which $\pi_1(\mathcal{T};x)$ acts via matrices of the form $\begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}$ where $\chi : \pi_1(\mathcal{T};x) \longrightarrow \mathbb{G}_a$ is a character.

This defines a map $\mathrm{Ext}^1_{\mathcal{T}}(\mathbb{1},\mathbb{1}) \to \mathrm{Hom}(\pi_1(\mathcal{T};x),\mathbb{G}_a)$, which is easily checked to be F-linear. It is injective, for if E gives rise to the trivial character, then $E_x = \mathbb{1} \oplus \mathbb{1}$ as $\pi_1(\mathcal{T};x)$-representations, so E is split an an extension in $\mathcal{T}$. For surjectivity, any character $\chi : \pi_1(\mathcal{T};x) \longrightarrow \mathbb{G}_a$ gives rise to an extension of π_1-representations and hence an extension in $\mathcal{T}$

We will later use the fact that universal objects in unipotent Tannakian categories admit a particularly explicit description.

<u>Definition</u>: Let $\mathcal{T}$ be a unipotent neutral Tannakian category and $\omega_x : \mathcal{T} \longrightarrow \underline{\mathrm{Vec}}_F$ a fibre functor. Assume for simplicity that $\mathrm{Ext}^1_{\mathcal{T}}(\mathbb{1}, \mathbb{1})$ is finite-dimensional.

Define a sequence $({}_x E^{\mathcal{T}}_i)_{i \geq 0}$ of objects of $\mathcal{T}$ recursively, by setting ${}_x E^{\mathcal{T}}_0 = \mathbb{1}$ and for $n+1$ letting ${}_x E^{\mathcal{T}}_{n+1}$ be the extension

$$0 \longrightarrow \mathbb{1} \otimes_F \mathrm{Ext}^1_{\mathcal{T}}({}_x E^{\mathcal{T}}_n, \mathbb{1})^* \longrightarrow {}_x E_{n+1} \longrightarrow {}_x E_n \longrightarrow 0$$

whose extension-class is the identity in

$$\mathrm{End}(\mathrm{Ext}^1_{\mathcal{T}}({}_x E^{\mathcal{T}}_n, \mathbb{1})^*) = \mathrm{Ext}^1_{\mathcal{T}}({}_x E_n, \mathbb{1}) \otimes \mathrm{Ext}^1_{\mathcal{T}}({}_x E_n, \mathbb{1})^*$$
$$= \mathrm{Ext}^1_{\mathcal{T}}({}_x E_n, \mathbb{1} \otimes_F \mathrm{Ext}^1_{\mathcal{T}}({}_x E^{\mathcal{T}}_n, \mathbb{1})^*).$$

Define elements $e^{\mathcal{T}}_{x,n} \in {}_x E^{\mathcal{T}}_{n,x}$ by $e^{\mathcal{T}}_{x,0} = 1 \in F = \mathbb{1}_x$ and thereafter letting $e^{\mathcal{T}}_{x,n+1}$ be any preimage of $e^{\mathcal{T}}_{x,n}$ in ${}_x E^{\mathcal{T}}_{n+1,x}$. We let ${}_x E^{\mathcal{T}} = \text{"}\varprojlim_n\text{"}\, {}_x E^{\mathcal{T}}_n \in \text{pro-}\mathcal{T}$ and $e^{\mathcal{T}}_x = (e^{\mathcal{T}}_{x,n}) \in {}_x E^{\mathcal{T}}_x$.

wrt the maps ${}_x E^{\mathcal{T}}_{n+1} \to {}_x E^{\mathcal{T}}_n$.

<u>Theorem</u>: $({}_x E^{\mathcal{T}}, e_x^{\mathcal{T}})$ pro-represents ω_x.

In fact, let $\mathcal{T}^{\leq n}$ denote the subcategory of objects of unipotency class $\leq n$, i.e. such that there exists a filtration

$$0 = \mathrm{Fil}_{-1} E \leq \mathrm{Fil}_0 E \leq \dots \leq \mathrm{Fil}_n E = E$$

whose graded pieces are each isomorphic to $\underline{\mathbb{1}}^{\oplus r_i}$ for some $r_i \geq 0$. Then $({}_x E_n^{\mathcal{T}}, e_{x,n}^{\mathcal{T}})$ represents the restriction of ω_x to $\mathcal{T}^{\leq n}$.

<u>Proof</u> (omitted in lecture)

We proceed inductively on n. When $n=0$, $\mathcal{T}^{\leq 0}$ is the category of objects isomorphic to a direct sum of copies of $\underline{\mathbb{1}}$, so is ~~isomorphic~~ equivalent to <u>Vec</u>$_F$ via the functor ω_x. Since $(F, 1)$ represents the identity functor on <u>Vec</u>$_F$, it follows that $({}_x E_0^{\mathcal{T}}, e_{x,0}^{\mathcal{T}})$ represents $\omega_x|_{\mathcal{T}^{\leq 0}}$.

Now suppose inductively that $({}_x E_n^{\mathcal{T}}, e_{x,n}^{\mathcal{T}})$ represents $\omega_x|_{\mathcal{T}^{\leq n}}$. Then certainly ${}_x E_{n+1}^{\mathcal{T}} \in \mathcal{T}^{\leq n+1}$.

If $E \in \mathcal{T}^{\leq n+1}$, then we can write E as an extension

$$0 \to \mathbb{1} \otimes V \to E \to E' \to 0 \quad (*)$$

for $E' \in \mathcal{T}^{\leq n}$ and $V \in \underline{\mathrm{Vec}}_F$. Let $e \in E_x$, and write $e' \in E'_x$ for its image. By inductive hypothesis, there exists a unique morphism $f' : {}_xE^{\mathcal{T}}_n \to E'$ s.t. $f'(e^{\mathcal{T}}_{x,n}) = e'$.

pulling back $*$ along f' yields an extension

$$0 \to \mathbb{1} \otimes V \to \hat{E} \to {}_xE^{\mathcal{T}}_n \to 0$$

Now by construction any extension of ${}_xE^{\mathcal{T}}_n$ splits when pulled back to ${}_xE^{\mathcal{T}}_{n+1}$. This means that there is a morphism $\hat{f} : {}_xE^{\mathcal{T}}_{n+1}$ fitting into a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{1} \otimes \mathrm{Ext}^1_{\mathcal{T}}({}_xE^{\mathcal{T}}_n, \mathbb{1})^* \arrow[r] \arrow[d] & {}_xE^{\mathcal{T}}_{n+1} \arrow[r] \arrow[d, "\hat{f}"] & {}_xE^{\mathcal{T}}_n \arrow[r] \arrow[d, "f'"] & 0 \\
0 \arrow[r] & \mathbb{1} \otimes V \arrow[r] & E \arrow[r] & E' \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Now let

$\partial = e - \hat{f}(e^{\mathcal{T}}_{x,n+1}) \in \ker(E_x \to E'_x) = V.$

and let $g : {}_x E^\alpha_{n+1} \to \mathbb{1} \otimes V$ be the composite

$${}_x E^\alpha_{n+1} \to {}_x E^\alpha_0 = \underline{\mathbb{1}} \xrightarrow{\delta} \mathbb{1} \otimes V$$

so $g(e^\alpha_{x,n+1}) = \delta$ also. Thus the map

$f := \tilde{f} + g : {}_x E^\alpha_{n+1} \to E$ satisfies $f({}_x e^\alpha_{n+1}) = e$.

It remains to show that f is the unique such map, or equivalently that if

$f : {}_x E^\alpha_{n+1} \to E$ is a map such that

$f(e^\alpha_{x,n+1}) = 0$ then $f = 0$.

Inducting over a unipotent filtration on E, it suffices to prove this in the case $E = \underline{\mathbb{1}}$.

But by construction the coboundary map

$$\mathrm{Hom}_\alpha(\underline{\mathbb{1}} \otimes \mathrm{Ext}^1_\alpha({}_x E^\alpha_n, \mathbb{1})^*, \mathbb{1}) \to \mathrm{Ext}^1_\alpha({}_x E^\alpha_n, \mathbb{1})$$

coming from the extension

$$0 \to \underline{\mathbb{1}} \otimes \mathrm{Ext}^1_\alpha({}_x E_n, \underline{\mathbb{1}})^* \to {}_x E^\alpha_{n+1} \to {}_x E^\alpha_n \to 0$$

is an isomorphism, so

$$\mathrm{Hom}_\alpha({}_x E^\alpha_{n+1}, \underline{\mathbb{1}}) = \mathrm{Hom}_\alpha({}_x E^\alpha_n, \underline{\mathbb{1}}).$$

Applying this recursively implies that any map $f : {}_x E^\alpha_{n+1} \to \underline{\mathbb{1}}$ factors through ${}_x E^\alpha_0 = \underline{\mathbb{1}}$. So if $f(e^\alpha_{x,n+1}) = 0$ then $f = 0$. $\square$.