# Lecture 4: Malčev completion

Now we're finally ready to define the $\mathbb{Q}_p$-pro-unipotent étale fundamental group(oid) of a smooth variety in characteristic 0. We will do this in two different ways. For today, our perspective is that we want to start from the profinite étale fundamental group and turn it into a $\mathbb{Q}_p$-pro-unipotent group in some universal way. Informally, it is informative to think of this construction as defining a "tensor product"

$$\mathbb{Q}_p \otimes_{\hat{\mathbb{Z}}} \pi_1^{\text{ét}}(X_{\bar{k}}; x)$$

Of course, this doesn't make sense when $\pi_1^{\text{ét}}(X_{\bar{k}}; x)$ is non-abelian. The way we make sense of this "non-abelian tensor product" is by an operation known as <u>Malčev completion</u>, which assigns to any finitely generated profinite group $\Pi$ a $\mathbb{Q}_p$-pro-unipotent group $\Pi_{\mathbb{Q}_p}$, with the property that the graded pieces of the descending central series of $\Pi_{\mathbb{Q}_p}$ are given by $\mathrm{gr}_\Gamma^\bullet \Pi_{\mathbb{Q}_p} = \mathbb{Q}_p \otimes_{\hat{\mathbb{Z}}} \mathrm{gr}_\Gamma^\bullet \Pi$.

is abelian

## § Mal'čev completion

We begin with some conventions on topologies. Let $F$ be a topological field of characteristic $0$. If $Z$ is an affine $F$-scheme of finite type, embedded as a closed subscheme of $\mathbb{A}^n_F$, then we endow $Z(F) \subseteq F^n$ with the subspace topology. Morphisms of affine $F$-schemes of finite type are all continuous in this topology, so in particular the topology on $Z(F)$ is independent of the choice of affine embedding. In general, if $Z$ is an affine $F$-scheme (not necessarily of finite type), then we can write $Z$ as a cofiltered limit of affine $F$-schemes of finite type $Z_i$, and endow $Z(F) = \varprojlim_i Z_i(F)$ with the inverse limit topology. This is again independent of any choices.

If $U$ is an affine group scheme, this makes $U(F)$ into a topological group.

Using this, we can define Mal'čev completion in a purely abstract way.

<u>Definition</u>: Let $F$ be a topological field of characteristic $0$ and $\Pi$ a topological group. Then the functor

$$\begin{aligned} \underline{\text{pro-Unipt}}_F &\longrightarrow \underline{\text{Set}} \\ U &\longmapsto \mathrm{Hom}_{cts}(\Pi, U(F)) \end{aligned}$$

is representable. We call the representing object the <u>Mal'čev completion</u> $\Pi_F$ of $\Pi$.

Explicitly, this means the following. The Mal'čev completion $\Pi_F$ is a pro-unipotent group over $F$ and comes with a continuous group homomorphism $\phi: \Pi \longrightarrow \Pi_F(F)$. This satisfies the following universal property: for any pro-unipotent group $U/F$ and any continuous homomorphism $f: \Pi \longrightarrow U(F)$, there exists a unique homomorphism $\hat{f}: \Pi_F \rightarrow U$ of pro-unipotent groups such that

$$\begin{tikzcd} \Pi \arrow[r, "\phi"] \arrow[rr, bend right, "f"'] & \Pi_F(F) \arrow[r, "\hat{f}"] & U(F) \end{tikzcd}$$

commutes.

Of course, we should justify why the Malčev completion exists. (Once it exists, it is unique up to unique isomorphism and functorial with respect to continuous group homomorphisms by the usual category-theoretic facts.) For this, since the category of pro-unipotent groups is the pro-category of unipotent groups, what we are wanting to show is that the functor

$$\begin{array}{ccc} \underline{\mathrm{Unipt}}_F & \longrightarrow & \underline{\mathrm{Set}} \\ U & \longmapsto & \mathrm{Hom}_{\mathrm{cts}}(\Pi, U(F)) \end{array}$$

is pro-representable. But this holds by the pro-representability theorem ($\underline{\mathrm{Unipt}}_F$ is essentially small, has finite limits and the functor in question preserves finite limits).

## § Explicit description of Malčev completion

There are a couple of ways to describe Malčev completion explicitly, of which we give just one. Suppose that $\Pi$ is a finitely generated profinite group, and that $F = \mathbb{Q}_p$. Define

$$\mathbb{Z}_p[\![\Pi]\!] = \varprojlim_N \mathbb{Z}_p[\Pi/N],$$

where $N$ ranges over all open normal subgroups of $\Pi$. $\mathbb{Z}_p[\![\Pi]\!]$ is a topological $\mathbb{Z}_p$-algebra, coming ~~with~~ with a continuous $\mathbb{Z}_p$-algebra homomorphism $\varepsilon : \mathbb{Z}_p[\![\Pi]\!] \longrightarrow \mathbb{Z}_p$ sending all elements of $\Pi$ to $1$. The kernel of $\varepsilon$ is called the <u>augmentation ideal</u> $I \trianglelefteq \mathbb{Z}_p[\![\Pi]\!]$, and we write $I^n$ for the closure of the $n^{\text{th}}$ power of $I$. Each quotient $\mathbb{Z}_p[\![\Pi]\!]/I^{n+1}$ is finitely generated as a $\mathbb{Z}_p$-module, so we can define a complete $\mathbb{Q}_p$-algebra

$$\mathbb{Q}_p[\![\Pi]\!] := \varprojlim_n \left( \mathbb{Q}_p \underset{\mathbb{Z}_p}{\otimes} \mathbb{Z}_p[\![\Pi]\!]/I^{n+1} \right)$$

**Proposition:** There is a natural comultiplication on $\mathbb{Q}_p[[\Pi]]$ making it into an $I$-adically complete cocommutative Hopf algebra, whose associated pro-unipotent group is the Mal'čev completion $\Pi_{\mathbb{Q}_p}$ of $\Pi$.

**Proof:** omitted.

## § The $\mathbb{Q}_p$-pro-unipotent étale fundamental groupoid

We give our first definition of the pro-unipotent étale fundamental groupoid.

<u>Definition</u>: Let $K$ be a field of characteristic $0$ with algebraic closure $\bar{K}$. ~~and~~ Let $X/K$ be a smooth variety, and let $x$ be a geometric point of $X_{\bar{K}}$. We define the $\mathbb{Q}_p$-pro-unipotent étale fundamental group of $(X_{\bar{K}}, x)$ to be

$$\pi_1^{\mathbb{Q}_p}(X_{\bar{K}}; x) := \pi_1^{\acute{e}t}(X_{\bar{K}}; x)_{\mathbb{Q}_p},$$

i.e. the $\mathbb{Q}_p$-Malčev completion of the profinite fundamental group. (There is a similar definition for groupoids; we omit it here.)

We can be quite explicit about the structure of the pro-unipotent group. For this, we need a lemma:

<u>Lemma</u>: Let $\Pi$ be a finitely generated discrete group, with profinite completion $\hat{\Pi}$. Then the natural map $\Pi_{\mathbb{Q}_p} \longrightarrow \hat{\Pi}_{\mathbb{Q}_p}$ is an isomorphism.

Proof: Unwinding definitions, the lemma amounts to the claim that for any $\mathbb{Q}_p$-pro-unipotent group $U$ and any homomorphism $\Pi \xrightarrow{f} U(\mathbb{Q}_p)$, there is a unique continuous group homomorphism $\hat{\Pi} \xrightarrow{\hat{f}} U(\mathbb{Q}_p)$ extending $f$. It suffices to verify this in the case $U = Un_m$. We define for $k \geq 0$ the subgroup $Un_m(p^{-k}\mathbb{Z}_p)$ of $Un_m(\mathbb{Q}_p)$ to consist of the matrizes of the form

$$\begin{pmatrix} 1 & p^{-k}* & p^{-2k}* & p^{-3k}* & \cdots \\ & 1 & p^{-k}* & p^{-2k}* & \cdots \\ & & 1 & p^{-k}* & \cdots \\ & 0 & & 1 & \ddots \\ & & & & 1 \end{pmatrix}$$

with all $*$'s in $\mathbb{Z}_p$. Each $Un_m(p^{-k}\mathbb{Z}_p)$ is profinite, and $Un_m(\mathbb{Q}_p) = \bigcup_k Un_m(p^{-k}\mathbb{Z}_p)$ with the induced topology. Any homomorphism $f : \Pi \to U(\mathbb{Q}_p)$ must factor through some $Un_m(p^{-k}\mathbb{Z}_p)$ and hence must factor uniquely through $\hat{\Pi}$ as $Un_m(p^{-k}\mathbb{Z}_p)$ is profinite. This is what we wanted to show $\square$

Here is how we will use the lemma. Suppose that $X/k$ is a smooth variety ($k$ of characteristic 0, as always) and choose some $x \in X(\bar{k})$ to use as a geometric basepoint. By the Lefschetz Principle, there is some subfield $k_0 \subseteq \bar{k}$ such that $X$ and $x$ are defined over $k_0$ and $k_0$ embeds in $\mathbb{C}$. So

$$
\begin{aligned}
\pi_1^{\mathbb{Q}_p}(X_{\bar{k}};x) &\cong \pi_1^{\text{ét}}(X_{\bar{k}};x)_{\mathbb{Q}_p} && \text{(definition)} \\
&\cong \pi_1^{\text{ét}}(X_{\mathbb{C}};x)_{\mathbb{Q}_p} && \text{(base change)} \\
&\cong \left(\hat{\pi}_1(X(\mathbb{C});x)\right)_{\mathbb{Q}_p} && \text{(Riemann existence)} \\
&\cong \pi_1(X(\mathbb{C});x)_{\mathbb{Q}_p} && \text{(lemma)}
\end{aligned}
$$

i.e. $\pi_1^{\mathbb{Q}_p}(X_{\bar{k}};x)$ is the $\mathbb{Q}_p$-Malcev completion of the topological fundamental group $\pi_1(X(\mathbb{C});x)$ with its discrete topology.

<u>Corollary</u>: $\pi_1^{\mathbb{Q}_p}(X_{\bar{k}};x)$ is topologically finitely generated.

<u>Corollary</u>: If $X/k$ is a smooth curve, which is the complement of a divisor $D$ of degree $r$ in a smooth projective curve $\bar{X}$ of genus $g$, then $\pi_1^{\mathbb{Q}_p}(X_{\bar{k}};x)$ is the $\mathbb{Q}_p$-pro-unipotent group on $2g+r$ generators subject to one relation $\prod_{i=1}^{g}[a_i,b_i]\cdot\prod_{i=1}^{r}c_i=$