Lecture 2: The profinite étale fundamental groupoid

In the 1960s, GROTHENDIECK became interested in adapting tools from algebraic topology to be used in the study of scheme theory. The two most famous examples are étale cohomology and étale fundamental groupoids, which mimic singular cohomology and fundamental groupoids in topology, respectively. In this lecture, we will review Grothendieck's construction of the profinite étale fundamental groupoid, as a first step towards constructing the $\mathbb{Q}_p$-pro-unipotent étale fundamental groupoid, a central object in non-abelian Chabauty.

Fundamental groupoids in topology

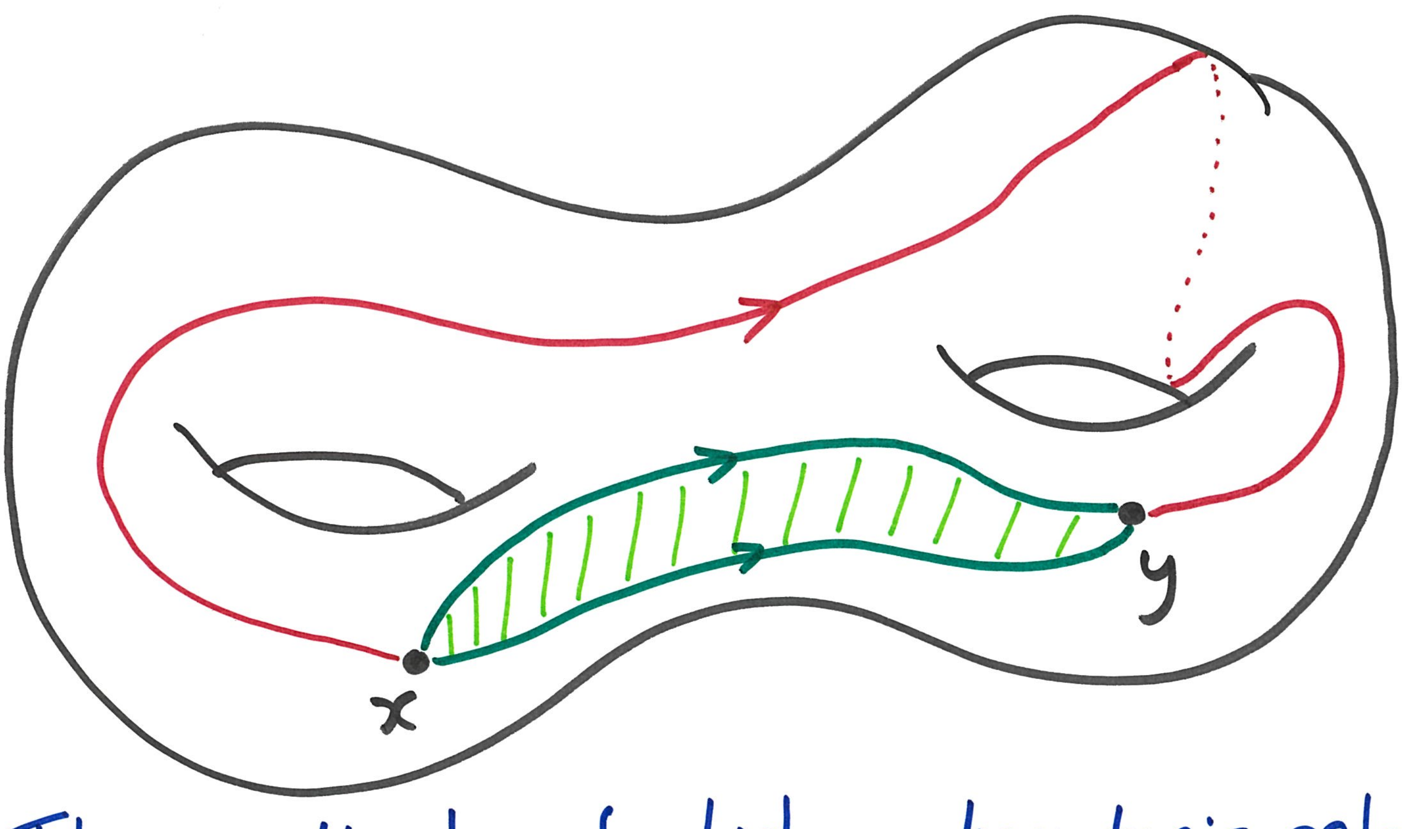
Let X be a locally path-connected, semilocally simply connected topological space, not necessarily connected.

Definition: Let $x, y \in X$.

i.e. $\gamma : [0,1] \to X$
$\gamma(0) = x$, $\gamma(1) = y$

$$\pi_1(X; x, y) = \{\text{continuous paths } \gamma : x \rightsquigarrow y\} \,/\, \text{homotopy relative to endpoints}$$

"path-set" or "path-torsor" (if non-empty)



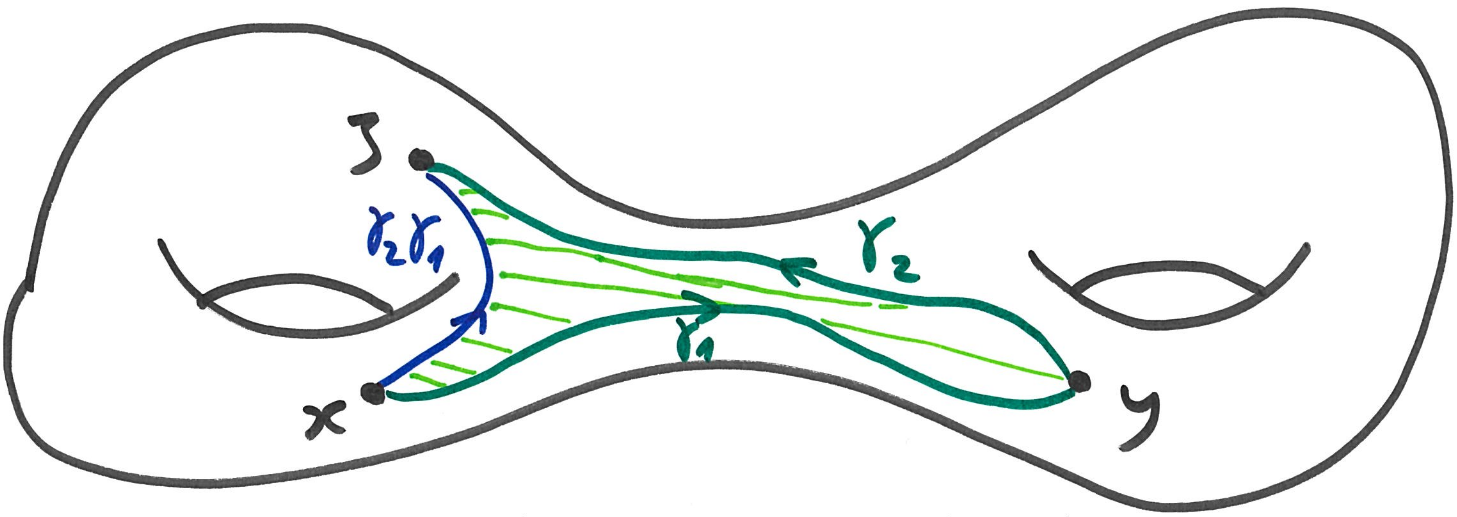
Three paths, two of which are homotopic relative to their endpoints.

Given paths $\gamma_1 : x \rightsquigarrow y$, $\gamma_2 : y \rightsquigarrow z$ we can form the composite $\gamma_2\gamma_1 : x \rightsquigarrow z$ "γ_1 then γ_2".

⚠ Topologists would denote this $\gamma_1\gamma_2$.

This induces a <u>composition map</u>

$$\pi_1(X; y, z) \times \pi_1(X; x, y) \longrightarrow \pi_1(X; x, y)$$



Also for $x \in X$ have an <u>identity path</u>

$$1_x \in \pi_1(X; x, x) \quad \text{"do nothing"}$$

And for $\gamma : x \rightsquigarrow y$ have an <u>inverse path</u> $\gamma^{-1} : y \rightsquigarrow x$ "go backwards along γ" defining a <u>reversal/inverse map</u>

$$\pi_1(X; x, y) \longrightarrow \pi_1(X; y, x).$$

These operations make the sets $\pi_1(X;x,y)$ look like a group, albeit one which is spread out over several points x.

<u>Definition</u> A <u>groupoid</u> Π consists of

- a set V of "<u>vertices</u>"
- for $x,y \in V$, a set $\Pi(x,y)$ of "<u>paths from x to y</u>"
- for $x,y,z \in V$, a <u>composition map</u>

$$\begin{aligned} \Pi(y,z) \times \Pi(x,y) &\longrightarrow \Pi(x,z) \\ (\gamma_2\ ,\ \gamma_1) &\longmapsto \gamma_2 \cdot \gamma_1 \end{aligned}$$

- for $x \in V$, an <u>identity</u>

$$1_x \in \Pi(x,x)$$

- for $x,y \in V$, a <u>reversal map</u>

$$\begin{aligned} \Pi(x,y) &\longrightarrow \Pi(y,x) \\ \gamma &\longmapsto \gamma^{-1} \end{aligned}$$

These are required to satisfy axioms

- (Associativity) $\gamma_3 \cdot (\gamma_2 \cdot \gamma_1) = (\gamma_3 \cdot \gamma_2) \cdot \gamma_1$ (for $\gamma_1 : w \rightsquigarrow x$, $\gamma_2 : x \rightsquigarrow y$, $\gamma_3 : y \rightsquigarrow z$)
- (Identities) $\gamma \cdot 1_x = \gamma = 1_y \cdot \gamma$ (for $\gamma : x \rightsquigarrow y$)
- (Inverses) $\gamma^{-1} \cdot \gamma = 1_x$ and $\gamma \cdot \gamma^{-1} = 1_y$ (for $\gamma : x \rightsquigarrow y$)

Remarks:

1. If $V=\{*\}$ is a singleton, then groupoids on vertex set V are the same thing as groups, via $\Pi \longmapsto \Pi(*,*)$. In general, if Π is a groupoid and $x \in V$, then

$$\Pi(x) := \Pi(x,x) \text{ is a group.}$$

2. For $x,y \in V$, either $\Pi(x,y)=\emptyset$ or $\Pi(x)$ and $\Pi(y)$ act simply transitively on $\Pi(x,y)$, from the right and left respectively, via the composition in Π.

3. If $\Pi(x,y) \neq \emptyset$, then $\Pi(x)$ is isomorphic to $\Pi(y)$ as groups, via $\gamma \longmapsto \gamma_0 \gamma \gamma_0^{-1}$ for a fixed $\gamma_0 \in \Pi(x,y)$.

⚠ Even though $\Pi(x)$ and $\Pi(y)$ are isomorphic, we don't want to pretend they are "the same", since the isomorphism is non-canonical.

4. For X as above, the sets $\pi_1(X;x,y)$ form a groupoid whose vertices are the points of X, the <u>fundamental groupoid</u> $\pi_1(X)$ of X.

Covering spaces and fundamental groupoids

Definition: A covering space of X is a continuous map $p: X' \longrightarrow X$ which, locally on X, is isomorphic to the projection

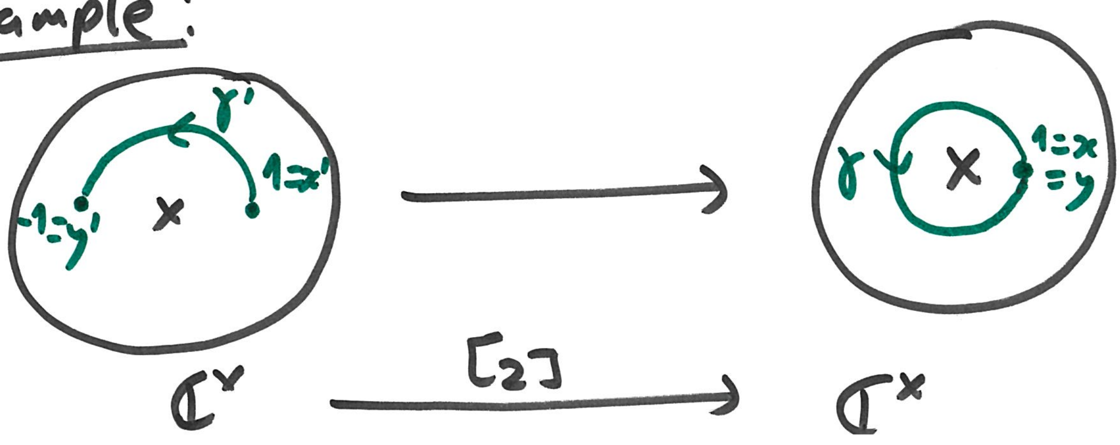
$$X \times (\text{discrete set}) \longrightarrow X$$

Examples: $\mathbb{C}^\times \xrightarrow{[2]} \mathbb{C}^\times$, $\quad \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$

Fact: If $p: X' \longrightarrow X$ is a covering space, $x, y \in X$, $\gamma : x \rightsquigarrow y$ ~~$\gamma \in \pi_1(X; x, y)$~~ and $x' \in X'_x := p^{-1}(x)$, then γ lifts to a path $\gamma' : x' \rightsquigarrow y'$ in X' for some $y' \in X'_y = p^{-1}(y)$. This defines a monodromy action

$$\begin{aligned} \pi_1(X; x, y) \times X'_x &\longrightarrow X'_y \\ (\gamma, x') &\longmapsto (\gamma \cdot x' := y') \end{aligned}$$

Example:



$\gamma \cdot 1 = -1$

These action maps are natural in the covering space X'.

<u>Definition</u>: A morphism from a covering space $p: X' \to X$ to a covering space $p': X'' \to X$ is a continuous map $f: X' \to X''$ making commute

$$\begin{tikzcd} X' \arrow[rr, "f"] \arrow[dr, "p"'] & & X'' \arrow[dl, "p'"] \\ & X & \end{tikzcd}.$$

Covering spaces + morphisms form a category $Cov(X)$. Any $x \in X$ determines a functor

$$\begin{aligned} \omega_x : Cov(X) &\longrightarrow Set \\ X' &\longmapsto X'_x \end{aligned}$$

"fibre functor"

Naturality of monodromy actions means they define a map

$$\begin{aligned} \pi_1(X; x, y) &\longrightarrow Iso(\omega_x, \omega_y) \quad \circledast \\ \gamma &\longmapsto \left(\text{Natural iso whose component at } X' \in Cov(X) \text{ is the monodromy action } \begin{aligned} X'_x &\xrightarrow{\sim} X'_y \\ x' &\mapsto \gamma \cdot x' \end{aligned} \right) \end{aligned}$$

The composite

$$\pi_1(X;x,y) \longrightarrow \mathrm{Hom}(\omega_x,\omega_y) \overset{\text{Yoneda Lemma}}{\cong} \tilde{X}_y$$

is given by $\gamma \longmapsto \underbrace{\gamma\cdot\tilde{x}}_{\text{monodromy action.}}$

But for all $\tilde{y} \in \tilde{X}_y$, there is a unique path $\tilde{\gamma} : \tilde{x} \rightsquigarrow \tilde{y}$ up to homotopy, as $\tilde{X}$ is simply connected. The image of $\tilde{\gamma}$ in $\pi_1(X;x,y)$ is the unique γ s.t. $\gamma\cdot\tilde{x} = \tilde{y}$. So

$$\pi_1(X;x,y) \longrightarrow \tilde{X}_y$$

is bijective and we are done. $\square$

The profinite étale fundamental groupoid

If X is a scheme, it is not obvious how to define its fundamental groupoid $\pi_1(X)$, as it's unclear how to define paths and homotopies in any sensible way. Grothendieck's insight was that since in topology we have

$$\pi_1(X; x, y) = \mathrm{Iso}(\omega_x, \omega_y) = \mathrm{Hom}(\omega_x, \omega_y),$$

we could use this as the definition of the fundamental groupoid just in terms of covering spaces. So to define the fundamental groupoid of a scheme X, we're looking for two things:

1. a category of "covering spaces" of X
2. for every point of x, an associated "fibre functor"

Definition: A finite étale covering of a scheme X is a finite étale morphism $p : X' \longrightarrow X$. These form a category $\text{FÉt}(X)$.

Example: If k is a field of characteristic $\neq 2$, then $[2] : \mathbb{G}_{m,k} \longrightarrow \mathbb{G}_{m,k}$ is finite étale

Theorem (RIEMANN existence) Suppose X is of finite type over $\mathbb{C}$. Then if $p : X' \longrightarrow X$ is finite étale, then $p : X'(\mathbb{C}) \longrightarrow X(\mathbb{C})$ is a covering space with finite fibres. This defines an equivalence of categories

$$\text{FÉt}(X) \xrightarrow{\sim} \text{Cov}(X(\mathbb{C}))_{\text{finite}}$$

N.B. $\exp : \mathbb{C} \longrightarrow \mathbb{C}^{\times}$ is not algebraic, so we have no hope of seeing this covering space in the world of schemes.

To define fibre functors, we need to be careful about the kinds of "points" we consider.

<u>Definition:</u> A <u>geometric point</u> of X is a morphism

$$\bar{x} : \operatorname{Spec}(\Omega) \longrightarrow X$$

of schemes, for some separably closed field Ω.
If $X' \longrightarrow X$ is finite étale, write $X'_{\bar{x}}$ for the set of morphisms ~~in~~ $\operatorname{Spec}(\Omega) \xrightarrow{\bar{x}'} X'$ such that

$$\begin{tikzcd} & X' \arrow[d] \\ \operatorname{Spec}(\Omega) \arrow[ur, "\bar{x}'"] \arrow[r, "\bar{x}"'] & X \end{tikzcd}$$

commutes.

<u>Fact:</u> X'_x is a finite set, of size $\deg_x(p)$.

<u>Example</u>: If X is finite type over a field k, k^s is a separable closure of k, then every k-point $x \in X(k)$ gives rise to a geometric point $\bar{x} : \operatorname{Spec}(k^s) \longrightarrow \operatorname{Spec}(k) \xrightarrow{x} X$.

We call the functor

$$\omega^{\text{ét}}_{\bar{x}} : \text{FÉt}(X) \longrightarrow \text{Set}_{\text{finite}}$$
$$X' \longmapsto X'_{\bar{x}}$$

the <u>fibre functor</u> at $\bar{x}$.

<u>Key definition</u> X a scheme, $\bar{x},\bar{y}$ geometric points

$$\pi_1^{ét}(X;\bar{x},\bar{y}) := \mathrm{Iso}(\omega_{\bar{x}}^{ét},\omega_{\bar{y}}^{ét}) \overset{\text{Fact: any nat. trans. is an iso}}{=} \mathrm{Hom}(\omega_{\bar{x}}^{ét},\omega_{\bar{y}}^{ét})$$

elements are called "paths" from $\bar{x}$ to $\bar{y}$.

Given geometric points $\bar{x},\bar{y},\bar{z}$, composition of natural isomorphisms / transformations defines a <u>composition map</u>

$$\pi_1^{ét}(X;\bar{y},\bar{z}) \times \pi_1^{ét}(X;\bar{x},\bar{y}) \longrightarrow \pi_1^{ét}(X;\bar{x},\bar{z}).$$

Together with the obvious identities and inverses, these make $\pi_1^{ét}(X;-,-)$ into a groupoid, the (<u>profinite</u>) <u>étale fundamental groupoid</u> of X.

Topology on the étale fundamental groupoid

The étale fundamental groupoid of a scheme X comes with an extra structure in the form of a topology.

Definition: If $\gamma_0 : X'_{\bar{x}} \xrightarrow{\sim} X'_{\bar{y}}$ is a bijection for some $X' \in \mathrm{F\acute{E}t}(X)$, we define

$$U_{\gamma_0} \subseteq \pi_1^{\text{ét}}(X; \bar{x}, \bar{y})$$

to be the set of natural isomorphisms $\gamma : \omega_{\bar{x}}^{\text{ét}} \xrightarrow{\sim} \omega_{\bar{y}}^{\text{ét}}$ such that $\gamma_{X'} = \gamma_0$.

We make $\pi_1^{\text{ét}}(X; \bar{x}, \bar{y})$ into a topological space by giving it the topology generated by the U_{γ_0}.

Equivalently, consider the two maps

$$F, G : \prod_{X'} \mathrm{Hom}(X'_{\bar{x}}, X'_{\bar{y}}) \longrightarrow \prod_{f : X'' \to X'} \mathrm{Hom}(X''_{\bar{x}}, X'_{\bar{y}})$$

where the fth component of $F((\gamma_{X'})_{X'})$ and $G((\gamma_{X'})_{X'})$ is $\gamma_{X'} \circ f_{\bar{x}}$ and $f_{\bar{y}} \circ \gamma_{X''}$ resp.

The set of natural transformations is by definition the equaliser

$$\begin{array}{l}\mathrm{Hom}(\omega^{\text{ét}}_{\bar{x}}, \omega^{\text{ét}}_{\bar{y}}) \longrightarrow \prod_{X'} \mathrm{Hom}(X'_{\bar{x}}, X'_{\bar{y}}) \underset{G}{\overset{F}{\rightrightarrows}} \prod_{f} \mathrm{Hom}(X''_{\bar{x}}, X'_{\bar{y}}) \\ \quad \| \\ \pi_1^{\text{ét}}(X; \bar{x}, \bar{y}) \end{array}$$

(cf. construction of natural transformations as an end)

We can give each of $\prod_{X'} \mathrm{Hom}(X'_{\bar{x}}, X'_{\bar{y}})$ and $\prod_{f} \mathrm{Hom}(X''_{\bar{x}}, X'_{\bar{y}})$ a natural topology as a product of finite sets, and this makes F and G continuous. The topology on $\pi_1^{\text{ét}}(X; \bar{x}, \bar{y})$ is the induced (subspace) topology on the equaliser.

<u>Proposition</u>: $\pi_1^{\text{ét}}(X; \bar{x}, \bar{y})$ is a profinite topological space (compact, Hausdorff, totally disconnected). The composition and inversion maps

$$\pi_1^{\text{ét}}(X; \bar{y}, \bar{z}) \times \pi_1^{\text{ét}}(X; \bar{x}, \bar{y}) \longrightarrow \pi_1^{\text{ét}}(X; \bar{x}, \bar{z})$$

$$\pi_1^{\text{ét}}(X; \bar{x}, \bar{y}) \longrightarrow \pi_1^{\text{ét}}(X; \bar{y}, \bar{z})$$

are continuous. In particular, $\pi_1^{\text{ét}}(X; \bar{x})$ is a profinite group for all geometric points $\bar{x}$.

Examples

1. $X = \mathrm{Spec}(k)$, k a field. A geometric point of X is the same as an embedding $k \hookrightarrow \Omega$ in a separably closed field.

$$\mathrm{F\acute{E}t}(X) = \{\underbrace{\text{finite étale } k\text{-algebras}}_{\text{finite products of finite separable extensions.}}\}^{op}$$

and the fibre functor is

$$L \longmapsto \mathrm{Hom}_{k\text{-algebras}}(L, \Omega)$$

$\Rightarrow \pi_1^{\text{ét}}(X; \bar{x}) \cong \mathrm{Gal}(k^s | k)$, where k^s is the separable closure of k inside Ω.

2. If Π is a group, its <u>profinite completion</u> is $\hat{\Pi} := \varprojlim_{N} (\Pi/N)$, where the inverse limit is taken over finite index normal subgroups N in Π, and $\hat{\Pi}$ is given the natural topology as an inverse limit of finite (discrete) sets.

More generally, if Π is a groupoid, we define

$$\hat{\Pi}(x,y) = \varprojlim_{N_x} \left(\Pi(x,y)/N_x \right) = \varprojlim_{N_y} \left(N_y \backslash \Pi(x,y) \right)$$

where the inverse limits are taken over finite index normal subgroups $N_x \trianglelefteq \Pi(x)$, $N_y \trianglelefteq \Pi(y)$.

<u>Theorem</u> Suppose X is of finite type over $\mathbb{C}$, $\bar{x}, \bar{y} \in X(\mathbb{C})$. Then there are homeomorphisms

$$\pi_1^{\text{ét}}(X; \bar{x}, \bar{y}) \cong \hat{\pi}_1(X(\mathbb{C}); \bar{x}, \bar{y})$$

compatible with composition, identities & inversion.

(Proof: Riemann existence.)

<u>Remark</u>: If K'/K is an extension of algebraically closed fields of characteristic 0, then $\pi_1^{\text{ét}}(X_{K'}) \cong \pi_1^{\text{ét}}(X)$ for all K-schemes X of finite type.

Universal covering

Like in topology, there is a good notion of the universal covering of a scheme X. Unlike in topology, the universal covering does not live in $\mathrm{F\acute{E}t}(X)$: finite étale coverings are too small. Instead, we need to enlarge $\mathrm{F\acute{E}t}(X)$.

Definition: A small category I is called *cofiltered* if I is non-empty and

1. $\forall i, j \in I$ $\exists k \in I$ and morphisms $k \to i$, $k \to j$
2. $\forall$ parallel arrows $f, g : j \to i$ in I, $\exists h : k \to j$ in I s.t. $f \circ h = g \circ h$.

$$\begin{tikzcd} & k \arrow[dl, dotted] \arrow[dr, dotted] & \\ i & & j \end{tikzcd} \qquad\qquad \begin{tikzcd} k \arrow[r, dotted, "h"] & j \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & i \end{tikzcd}$$

1. 2.

A *cofiltered limit* in a category $\mathcal{C}$ is a limit over a (small) cofiltered diagram.

Example: A diagram of shape

$$\cdots \to X_2 \to X_1 \to X_0$$

is cofiltered.

The pro-category of a category $\mathcal{C}$ is the category you get by (freely) adjoining cofiltered limits to $\mathcal{C}$.

<u>Definition:</u> Let $\mathcal{C}$ be a locally small category. pro-$\mathcal{C}$ is the category whose objects are cofiltered diagrams in $\mathcal{C}$, written as $(X_i)_{i\in I}$, or more suggestively $\text{``}\varprojlim_{i\in I}\text{''} X_i$.

The morphisms in pro-$\mathcal{C}$ are defined by

$$\mathrm{Hom}_{\text{pro-}\mathcal{C}}\left(\text{``}\varprojlim_{i\in I}\text{''} X_i, \text{``}\varprojlim_{j\in J}\text{''} Y_j\right) := \varprojlim_{j\in J} \varinjlim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j)$$

We omit the definition of composition of morphisms in pro-$\mathcal{C}$; it is the "obvious" definition once you unpack what the above notation means.

Proposition:

1. $\mathcal{C}$ embeds as a full subcategory of pro-$\mathcal{C}$, by thinking of every object of $\mathcal{C}$ as a one-object diagram.
2. pro-$\mathcal{C}$ has cofiltered limits; if $\mathcal{C}$ has finite limits, then pro-$\mathcal{C}$ has small limits.

Definition: Let $\mathcal{C}$ be locally small. A functor $F: \mathcal{C} \longrightarrow \mathrm{Set}$ is pro-representable if, equivalently

1. F is a (small) $\underbrace{\text{filtered colimit}}_{\text{dual notion to cofiltered limits}}$ of representable functors

$\Updownarrow$

2. The extension of F to a functor

$$F: \text{pro-}\mathcal{C} \longrightarrow \mathrm{Set}$$
$$F(\text{“}\varprojlim_{i\in I}\text{''} X_i) := \varprojlim_{i \in I} F(X_i)$$

is representable.

$\Updownarrow$

3. There is a cofiltered diagram $(X_i)_{i\in I}$ in $\mathcal{C}$ and a compatible system of elements $x_i \in F(X_i)$ s.t. for any $Y \in \mathcal{C}$ and $y \in F(Y)$, there exists some $i \in I$ and $f_i : X_i \longrightarrow Y$ in $\mathcal{C}$ s.t. $f_i(x_i) = y$, and (i, f_i) is unique "up to equivalence"

"Fundamental groupoid of a connected scheme is connected"

Corollary If X is a connected scheme, then $\pi_1^{\text{ét}}(X;\bar{x},\bar{y}) \neq \emptyset$ for all geometric points $\bar{x},\bar{y}$.

Proof: Let $(\tilde{X},\tilde{\bar{x}})$ be a universal covering at $\bar{x}$ and write $\tilde{X} = \text{"}\varprojlim_{i\in I}\text{"}\, \tilde{X}_i$ for finite étale coverings $\tilde{X}_i \to X$. So

$$\pi_1^{\text{ét}}(X;\bar{x},\bar{y}) = \mathrm{Hom}(\omega_{\bar{x}}^{\text{ét}},\omega_{\bar{y}}^{\text{ét}}) \overset{\text{Yoneda}}{\cong} \tilde{X}_{\bar{y}} = \varprojlim_{i\in I} \tilde{X}_{i,\bar{y}}$$

Now the image of $\tilde{X}_i \to X$ is closed and open and contains $\bar{x}$ for all i, so it is surjective and hence $\tilde{X}_{i,\bar{y}} \neq \emptyset$. ~~Now~~ But a cofiltered limit of non-empty finite sets is non-empty (MITTAG-LEFFLER), so

$$\pi_1^{\text{ét}}(X;\bar{x},\bar{y}) \cong \varprojlim_{i\in I} \tilde{X}_{i,\bar{y}} \neq \emptyset. \qquad \square$$